Proceedings of the
47th IEEE Conference on Decision and Control
Cancun, Mexico, Dec. 9-11, 2008

Robustness to Agent Loss in Vehicle Formations & Sensor Networks

Tyler H. Summers, Changbin Yu, and Brian D.O. Anderson
The Australian National University and National ICT Australia, Canberra, Australia
{tyler.summers,brad.yu,brian.anderson}@anu.edu.au

***Abstract*— A primary motivation for using large-scale vehicle formations and sensor networks is potential robustness to loss of a single agent or a small number of agents. In this paper, we address the problem of agent loss by introducing redundancy into the information architecture such that limited agent loss does not destroy desirable properties. We model the information architecture as a graph $G(V,E)$, where V is a set of vertices representing the agents and E is a set of edges representing information flow amongst the agents. We focus on two properties of the graph called rigidity and global rigidity, which are required for formation shape maintenance and sensor network self-localization, respectively. In particular, our objective in this paper is to investigate the structure of graphs in the plane with the property that rigidity or global rigidity is preserved after removing any single vertex (we call the property *2-vertex-rigidity* or *2-vertex-global-rigidity*, respectively). Information architectures with such properties would allow critical tasks, such as formation shape maintenance or self-localization, to be performed even in the event of agent failure. We review a characterization of a particular class of *2-vertex-rigidity* and develop a separate class, making significant strides toward a complete characterization. We also present for the first time a characterization of a particular class of *2-vertex-global-rigidity*. Finally, we list several related open problems and suggest directions for further research.**

I. INTRODUCTION

Autonomous vehicle formations and sensor networks are being progressively deployed to perform a variety of tasks including military reconnaissance and surveillance missions, environmental monitoring, and underwater exploration. Interest in these applications is reflected by the considerable attention from the literature [1]–[7]. We consider vehicle formations and sensor networks as collections of agents, each with sensing, communication, and computation capabilities, that work together to accomplish a task.

A primary motivation for using large-scale vehicle formations and sensor networks is robustness to loss of a single agent or a small number of agents. The loss could occur in a number of ways: due to enemy attack or jamming; due to random mechanical or electrical failure; or due to intentionally deploying an agent for a separate task. Further, large-scale formations and sensor networks are typically composed of relatively inexpensive agents that may be prone to such failures.

One could consider addressing agent loss in two separate ways: (1) perform a "self-repair" operation in the event of agent loss to recover desirable properties, or (2) introduce redundancy into the information architecture *a priori* such that agent loss does not destroy desirable properties. The "self-repair" approach is *reactive* in that the formation reacts to an agent loss event. In contrast, the redundancy approach is *proactive* in that redundancy is built into the formation *a priori* in anticipation of an agent loss event. The "self-repair" approach is addressed in [8]; in this paper, we consider the redundancy approach.

We model the information architecture with a graph $G(V,E)$, where V is a set of vertices representing agents and E is a set of edges representing information flow amongst the agents. We focus on two properties, called rigidity and global rigidity, which are required for formation shape maintenance and self-localization tasks, respectively, and have received significant attention recently in the literature (see e.g. [1]–[3], [6], [7], [9]). The formation shape maintenance task is to maintain all inter-agent distances constant such that the formation moves as a cohesive whole. In this case, the edge set E represents the set of inter-agent distances to be actively held constant via control of individual vehicle motion. If a suitably large and well-chosen set of inter-agent distances is held constant, then all remaining inter-agent distances will be constant as a consequence, thus maintaining formation shape. The self-localization task is to uniquely determine positions for each agent from knowledge of a partial set of inter-agent distances and knowledge of the positions in a global coordinate basis of several agents ("anchors"). In this case, the edge set represents the set of known inter-agent distances. Again, if a suitably large and well-chosen set of inter-agent distances is known, then the remaining inter-agent distances may be uniquely determined. Further, if the positions of three non-collinear agents are also known, then all other agent positions may be uniquely determined. These ideas are further explained in the next section.

The objective of this paper is to investigate the structure of graphs with the property that rigidity or global rigidity is preserved after removing any single vertex, which we call *2-vertex-rigidity*[1] or *2-vertex-global-rigidity*, respectively. Information architectures with these properties would allow critical tasks, such as formation shape maintenance or self-localization, to be performed even in the event of loss of any single agent. Currently, there are some subtle graph theoretic features that make obtaining a complete theoretic characterization of these properties difficult. However, we are motivated to work toward a complete characterization because it will be fundamental for further theoretic and practical developments, such as securing robustness to loss of more than one agent or developing efficient algorithms for determining whether a graph is *2-vertex-rigid* or *2-vertex-globally-rigid*.

[1] The reader may wonder why we use the term *2-vertex-rigidity*, rather than *1-vertex-rigidity*. As explained later, these definitions are analogous to the standard definitions for graph connectivity.

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In the main part of the paper, we overview general concepts in redundant rigidity, drawing from developments in the characterization of global rigidity due to Hendrickson, Jackson, and Jordan [10], [11] and the recent work by Yu and Anderson in [12]. We review the characterization of a particular class of *2-vertex-rigidity*, which we call *strongly minimal*, given by Servatius in [13]. The original contributions of the paper are as follows: we develop a separate class of *2-vertex-rigidity*, which we call *weakly minimal*, which makes significant strides toward a complete characterization. We also present for the first time a characterization of what we call *strongly minimal 2-vertex-global-rigidity*. The results here are for formations and sensor networks in two dimensions.

The paper is organized as follows: In Section 2 we present some standard graph theoretic concepts used to model information architectures in vehicle formations and sensor networks. We focus on describing the properties of rigidity and global rigidity. In Section 3, we review redundant rigidity concepts, particularly the work on *strongly minimal 2-vertex-rigidity* in [13], and present new developments for *weakly minimal 2-vertex-rigidity*. In Section 4, we present a complete characterization of *strongly minimal 2-vertex-global-rigidity*. Finally, Section 5 provides some summarizing and concluding remarks and identifies a few possible problems for future consideration.

II. BACKGROUND

In this section, we summarize the graph theoretic concepts used to model information architectures in vehicle formations and sensor networks. We describe rigidity and global rigidity, which are graph theoretic properties required to perform formation shape maintenance and self-localization, respectively.

A. Rigid Graph Theory

A fundamental task for vehicle formations is maintaining some prescribed geometric shape. Drawing from long-standing traditions in structural engineering and combinatorics (see e.g. [14] and references therein), rigid graph theory has been recently introduced in [1]–[3] as a means for describing the information architecture required to maintain formation shape.

A graph $G(V,E)$, where V is a set of vertices and $E \subseteq V \times V$ is a set of edges, provides a useful high-level model of the information architecture. We begin by formally defining a formation $F(G,p)$ as a graph G along with a mapping $p: V \rightarrow \Re^{d|V|}$ that assigns to each vertex a position in d-dimensional Euclidean space. Each agent is abstracted as a vertex in the graph. An edge is present between two vertices in the graph, or equivalently a link is present in the formation, whenever there is active maintenance of the Euclidean distance between the two agents. If a vertex v_j is connected by an edge to v_i, we call v_j a *neighbour* of v_i. The distance is maintained using a control law to govern the motion of one or both agents. Obviously, some appropriate quantities must be sensed for use in controlling distances to neighbours. Specifically, in virtually all formation control algorithms based on distance maintenance, each agent needs to sense relative positions to its neighbours in an arbitrary local coordinate basis. This involves sensing both distances to neighbours and angles to neighbours from some local reference. Alternatively, one could sense distances to neighbours and distances between pairs of neighbours and use the cosine law to obtain the appropriate angles.

Roughly speaking, a rigid formation is one that preserves its shape during a smooth motion, i.e. the distance between every pair of agents remains constant. Consider the function $f: \Re^{d|V|} \rightarrow \Re^{|E|}$ defined by

$$f(p) = [..., ||(p_i - p_j)||^2, ...] \qquad (1)$$

where the kth entry of f corresponds to the squared distance between vertices i and j when they are connected by an edge. Now, suppose the formation moves but $f(p)$ stays constant, i.e. rhw edges in E correspond to links where distance is preserved. Then expanding $f(p)$ about the constant value in a Taylor series and ignoring higher order terms, we obtain

$$J_f(p)\delta p = 0 \qquad (2)$$

where δp is an infinitesimal perturbation of the formation, and $J_f(p)$ is the Jacobian of f. This Jacobian is known as the *rigidity matrix* $R(p)$. Equivalently,

$$J_f(p)\dot{p} = 0 \qquad (3)$$

for a formation undergoing smooth motion. When the formation is rigid, the only permissible smooth motions are translation or rotation of the whole formation. In d dimensions, this accounts for $(1/2)d(d+1)$ linearly independent vectors. Thus the kernel of $J_f(p)$ has dimension $(1/2)d(d+1)$. This leads us to the following linear algebraic characterization of rigidity:

Theorem 1: A formation $F(G,p)$ is rigid[2] if and only if $rank[R(p)] = d|V| - d(d+1)/2$, which is the maximum rank $R(p)$ can have.

Because the rigidity matrix is a Jacobian of a rational function, it has the same rank for all points but a set of measure zero (via a nontrivial result of Sard [15]), corresponding to special vertex configurations (e.g. a set of agents are collinear or occupy the same point) that cause the rank deficiency. This leads to the notion of *generic* rigidity. For *generic* configurations (when the special configurations are precluded), information about formation rigidity is contained in the graph, allowing for the drawing of a purely combinatorial consequence of rigidity. The following theorem is presented as the "Necessary Counts Theorem" in [16]:

Theorem 2: If a graph $G(V,E)$ in $\Re^d$ is rigid, then there exists a subset E' of edges such that the induced subgraph $G'(V,E')$ satisfies the following:

- $|E'| = d|V| - d(d+1)/2$
- Any subgraph $G''(V'',E'')$ of G' with at least d vertices satisfies $|E''| \leq d|V''| - d(d+1)/2$.

[2] Actually, the term infinitesimally rigid is sometimes used. See [14] for further discussion of different rigidity concepts.

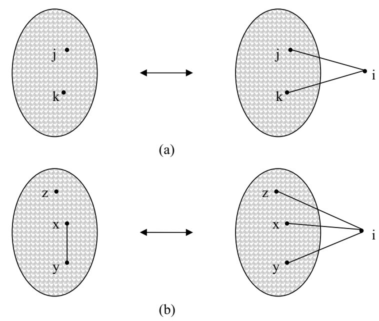

Fig. 1. Representation of (a) vertex addition operation (b) edge splitting operation.

A graph is called *minimally rigid* if it is rigid and there exists no rigid graph with the same number of vertices and a smaller number of edges. Equivalently, a graph is *minimally rigid* if removing any edge results in loss of rigidity. Intuitively, a minimally rigid graph on a prescribed vertex set must have a minimum number of edges, and the edges must be properly distributed. The set of basis edges corresponding to E' in Theorem 2 are called *independent*. An edge added to a graph is called *dependent* whenever the addition results in a subgraph that violates the second condition in Theorem 2. Equivalently, an edge is called *dependent* when the corresponding row of the rigidity matrix associated with a generic formation connected with the graph is linearly dependent on rows of the matrix present before addition of the edge. When a graph remains rigid after removing any edge, it is called *redundantly rigid*.

In the plane, we have a complete characterization of rigidity due to Laman [17]. In particular, the conditions given in Theorem 2 are both necessary and sufficient for rigidity. Further, there is a set of two basic operations, called Henneberg operations, that allow one to "grow" *every* minimally rigid graph in the plane from the complete graph on two vertices [14], [18]. Let j and k be two distinct vertices of a minimally rigid graph $G(V,E)$. A *vertex addition* operation involves adding a vertex i and edges ij and ik. Let x, y, and z be three distinct vertices of a minimally rigid graph with edge xy. An *edge splitting* operation involves removing xy and adding a vertex w and edges wx, wy, wz. The operations are illustrated in Figure 1.

A fundamental task for sensor networks is to determine uniquely the position of each agent from knowledge of certain inter-agent distances and the positions of a small number of agents. This task is related to a further concept called *global rigidity*. A graph in the plane is called *globally rigid* if two formations having the same inter-agent distances differ at most by translation, rotation, and reflection. In [7], Aspnes et al show that global rigidity is required for unique self-localization in sensor networks (and when the positions of any three non-collinear agents are known, global rigidity is sufficient for localizability of every agent). In [11], Jackson and Jordan prove a conjecture posed by Hendrickson in [10] that provides a complete characterization of global rigidity in the plane. The result is as follows:

Theorem 3: A graph $G(V,E)$ in the plane is globally rigid if and only if it is 3-vertex-connected and redundantly rigid.

A graph is called *minimally globally rigid* if it is globally rigid and there exists no globally rigid graph with the same number of vertices and a smaller number of edges, or equivalently, if removing any edge results in loss of global rigidity. This equivalency is related to the fact that the edge splitting operation described above can be used to "grow" *all* minimally globally rigid graphs from the complete graph on four vertices [19].

We are interested in graphs that preserve rigidity or global rigidity so that formation shape maintenance or self-localization can be performed even in the event of agent loss. We would like to find a complete characterization and similar operations that would allow one to "grow" such graphs.

III. REDUNDANT INFORMATION ARCHITECTURES

In this section, we investigate the structure of graphs in the plane with the property that rigidity is preserved when any single vertex and its incident edges are removed. Information architectures with this structure would allow vehicle formations to maintain formation shape even in the event of agent loss. We review general redundant rigidity definitions and concepts, and overview characterization of a particular class of *2-vertex-rigidity*, which we call *strongly minimal*, given by Servatius in [13]. We then present new developments of a particular class of *2-vertex-rigidity*, which we call *weakly minimal*, making significant strides toward a complete characterization.

A. REDUNDANT RIGIDITY CONCEPTS

In [13], Servatius introduced the notions of *edge birigidity* and *vertex birigidity* of a graph. A graph is called *edge birigid* if it remains rigid after removing any edge. As discussed previously, this property is used to characterize global rigidity (it is simply called *redundant rigidity* in that context). A graph is called *vertex birigid* if the graph remains rigid after removing any vertex and its incident edges.

In [12], Yu and Anderson introduced the generalized terms *k-edge-rigidity* and *k-vertex-rigidity*. A graph is called *k-edge-rigid* if it remains rigid after removing any $k-1$ edge(s). Similarly, a graph is called *k-vertex-rigid* if it remains rigid after removing any $k-1$ vertices. (For consistency of the results, it is convenient to define any graph with fewer than 3 vertices in two dimensions as nonrigid.) This notation is analogous to the standard notation for connectivity: a graph is k-edge-connected (k-vertex-connected) if it remains connected after removing any $k-1$ edges (vertices). In [12], Yu and Anderson characterize the relationship between *k-edge-rigidity* and *k-vertex-rigidity*, confirming the natural intuition that vertex-rigidity is a more demanding concept than edge-rigidity. They also explore redundant rigidity properties for special types of graphs, including the complete graph, the wheel graph, and powers of a graph. We shall use the terminology of [12] and restrict our attention in this paper to *2-vertex-rigidity*.

Remark 1: A graph is more vulnerable to loss of a vertex with large degree than a vertex with smaller degree because more edges are removed. For example, consider the wheel graph on n vertices, which consists of a single vertex (the "hub") connected to every vertex in a $(n-1)$-cycle. If any vertex in the cycle is removed, the graph remains rigid. However, if the hub is removed, the resulting graph is not rigid, and one must add $n-3$ new edges to recover rigidity. The hub is in a sense more important than the other vertices since the health of the formation is more vulnerable to its failure than to failure of any other vertex.

B. STRONGLY MINIMAL 2-VERTEX-RIGIDITY

It is natural to seek a characterization of *minimal 2-vertex rigidity* analogous to the way that a graph being rigid implies existence of a minimally rigid subgraph. However, as we will show in the following, the concept of minimality in 2-vertex-rigid graphs gives rise to a subtlety that contrasts with minimality in rigid graphs. Given a rigid graph $G(V,E)$, minimality is characterized equivalently by either of the following statements: (1) G has the minimum possible number of edges ($|E| = 2|V|-3$), and (2) removing any edge destroys rigidity. For 2-vertex-rigid graphs, an attempt to generalize these statements gives rise to two distinct types minimally 2-vertex-rigid graphs, which we call *strongly minimal* and *weakly minimal*:

- A 2-vertex-rigid graph is called *strongly minimal* if it has the minimum possible number of edges on a given number of vertices.
- A 2-vertex-rigid graph is called *weakly minimal* if it has more than the minimum possible number of edges on a given number of vertices, but has the property that removing any edge destroys 2-vertex-rigidity.

In [13], Servatius provides a characterization of strongly minimal 2-vertex-rigidity and gives an example that shows existence of a weakly minimal 2-vertex-rigid graph, which we now overview.

We begin with the following result, which gives a lower bound on the number of edges in a 2-vertex-rigid graph in terms of the number of vertices.

Lemma 1: If $G(V,E)$ is a 2-vertex-rigid graph on 5 or more vertices, then $|E| \geq 2|V|-1$.

Proof: Suppose $G(V,E)$ is a 2-vertex-rigid graph on 5 or more vertices with $|E| = 2|V|-2$. The average degree in a graph on $|V|$ vertices with $|E| = 2|V|-2$ is $4-4/|V|$. Thus, such a graph on 5 or more vertices has a vertex of degree at least 4. Removing such a vertex results in a graph $G'(V',E')$ where $|E'| = 2|V'|-4$. Thus, G' cannot be rigid, which contradicts our original assumption that G was 2-vertex-rigid. ■

Servatius uses the concept of *excess* to distinguish between our terms strongly and weakly minimal. The *excess* of a rigid graph $G(V,E)$ in 2 dimensions is defined as $|E|-(2|V|-3)$. A minimally rigid graph has excess zero (i.e. $|E| = 2|V|-3$).

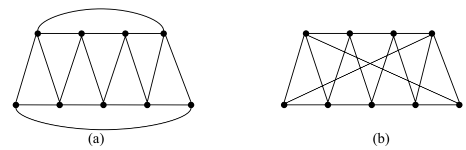

Fig. 2. Examples of the two possible partitions of the edge set for strongly minimal 2-vertex-rigid graphs: (a) the degree three vertices are adjacent, and (b) the degree three vertices are non-adjacent.

A minimally globally rigid graph has excess one (i.e. $|E| = 2|V|-2$). On 4 or fewer vertices, it is impossible to have $|E| \geq 2|V|-1$, and a graph must be complete to be 2-vertex-rigid. A strongly minimal graph (on 5 or more vertices) has excess two while a weakly minimal graph has excess more than two.

The following two results from [13] give a complete characterization of the structure of strongly minimal 2-vertex-rigid graphs.

Theorem 4: Let $G(V,E)$ be a strongly minimal 2-vertex-rigid graph on 5 or more vertices. Then G has exactly two vertices with degree 3 and the remaining have degree 4, which implies $|E| = 2|V|-1$.

Theorem 5: A graph $G(V,E)$ is strongly minimal 2-vertex-rigid if and only if G has exactly two vertices of degree 3 and there is a partition of the edge set E

$$E = E_1 \cup E_2 \cup \cdots \cup E_k$$

such that the graph induced by $E \setminus E_i$ is minimally redundantly rigid for all i, where either

- E_1 and E_2 are the edges incident to the two non-adjacent vertices of degree 3, respectively, and E_i is a single edge for $3 \leq i \leq k$
- E_1 is the union of the edges incident to the two adjacent vertices of degree 3, and E_i is a single edge for $2 \leq i \leq k$.

This can be thought of as a Laman-type characterization, analogous to minimal rigidity: there must be a minimum number of edges ($|E| = 2|V|-1$), and the edges must be properly distributed, as described in the theorem conditions. The two possible partitions of the edge set correspond to whether or not the two degree 3 vertices are adjacent.

Figure 2, originally from [13], shows examples of strongly minimal 2-vertex-rigid graphs with each type of partition. Note that triangles could be "stacked" together in such a fashion to produce arbitrarily large 2-vertex-rigid graphs of excess two.

In fact, Servatius also provides a way to "grow" all strongly minimal 2-vertex-rigid graphs using an operation similar to the edge splitting operation discussed previously. This operation gives an increase of $|V|$ by 1 and an increase of $|E|$ by 2, and thus preserves the constraint $|E| = 2|V|-1$. Thus, we have a complete characterization and a way to obtain all strongly minimal 2-vertex-rigid graphs.

C. WEAKLY MINIMAL 2-VERTEX-RIGIDITY

Can one simply add edges to a strongly minimal 2-vertex-rigid graph to obtain every (non-minimal) 2-vertex-rigid graph

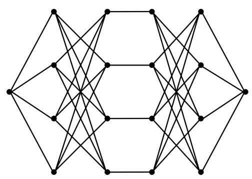

Fig. 3. Example of a weakly minimal 2-vertex-rigid graph. The graph has excess three, but removing any edge destroys 2-vertex-rigidity. Thus, this graph cannot be obtained by adding an edge to a graph described by Theorem 5.

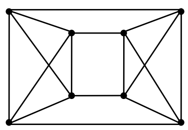

Fig. 4. Smallest example of a weakly minimal 2-vertex-rigid graph in the class defined by the Servatius example.

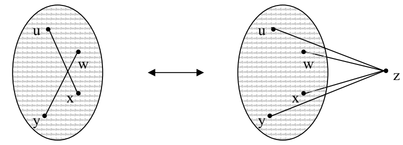

Fig. 5. Representation of the *X-replacement* operation.

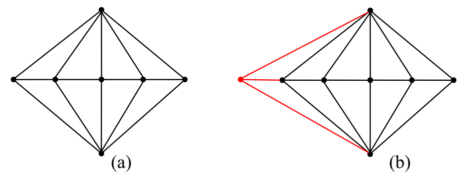

Fig. 6. A new class of weakly minimal 2-vertex-rigidity: (a) a weakly minimal 2-vertex-rigid graph with excess three, (b) adding a degree 3 vertex to create a weakly minimal 2-vertex-rigid graph with excess four. By successively adding a degree 3 vertex to either end, one can obtain weakly minimal 2-vertex-rigid graphs with arbitrarily large excess.

(as can be done for rigid graphs)? The answer to this question is no. The example in Figure 3, originally shown in [13], shows existence of a weakly minimal 2-vertex-rigid graph, which contains more than the minimum possible number of edges yet has the property that removing any edge destroys 2-vertex-rigidity.

Proposition 1: The graph $G(V,E)$ in Figure 3 is weakly minimal 2-vertex-rigid.

Proof: Every vertex has degree 4; thus, $|E| = 2|V|$, which is excess three. Observe that the subgraphs induced by the left and right nine vertices (call them G_L and G_R) both have excess one. Removing any vertex in G results in a graph of excess one that has exactly one subgraph of excess one, which is rigid. Thus, the graph is 2-vertex-rigid.

Now remove any edge outside of G_L; call the new graph G'. Then remove any degree 4 vertex in $G' \setminus G_L$. The resulting graph, call it G'', has an excess of zero with a subgraph of excess one (viz G_L), and so G'' is not rigid by Laman's theorem. Therefore, G' is not 2-vertex-rigid. Obviously, the same argument applies if we remove any edge outside of G_R. Hence, removing any edge in G destroys 2-vertex-rigidity, and thus G is weakly minimal 2-vertex-rigid. ■

This example points to a particular class of weakly minimal 2-vertex-rigid graphs that generally consist of two redundantly rigid subgraphs connected by four edges. The smallest such graph, presented here for the first time in Figure 4, consists of two complete graphs on four vertices connected by four edges. Further, we can use what is called the *X-replacement* operation, which is shown to preserve rigidity and redundant rigidity in [20], to "grow" arbitrarily large weakly minimal 2-vertex-rigid graphs. The operation is illustrated in Figure 5 and described below.

Definition 1: Given two non-adjacent edges ux and wy in a graph $G(V,E)$, an *X-replacement* adds a degree 4 vertex z to construct the graph $G'(V',E')$, where $V' = V \cup \{z\}$ and $E' = E \setminus \{ux, wy\} \cup \{uz, wz, xz, yz\}$.

Since the *X-replacement* preserves redundant rigidity, it can be applied successively to each redundantly rigid subgraph in Figure 4 (that is, each complete subgraph on four vertices) to create a class of weakly minimal 2-vertex-rigid graphs with excess three, which includes the Servatius example in Figure 3. Indeed, one can easily verify that the graph in Figure 4 can be obtained by repeatedly applying the reverse *X-replacement* operation on the left and right subgraphs in Figure 3.

In [13], Servatius poses an open question regarding existence of other weakly minimal 2-vertex-rigid graphs with larger excess. We have discovered such a new class that can have arbitrarily large excess. The graph shown in Figure 6(a) illustrates an example with excess three, and the graph shown in Figure 6(b) illustrates an example with excess four obtained by applying what we call a *degree 3 vertex addition*. Let i, j, and k be three distinct vertices in a graph $G(V,E)$. A *degree 3 vertex addition* operation adds a vertex l and edges il, jl, and kl. This operation preserves weakly minimal 2-vertex-rigidity under certain conditions given in [13] and also increases the excess by one. By successively applying the operation as shown, one can obtain weakly minimal 2-vertex-rigid graphs with arbitrarily large excess.

Proposition 2: The class of graphs illustrated in Figure 6 is weakly minimal 2-vertex-rigid.

Proof: Let $G(V,E)$ be the graph in Figure 6(a). One can easily verify that $|E| = 2|V|$ (excess three) and that removing any vertex from G results in a rigid graph; thus, G is 2-vertex-rigid. Now, remove any edge not incident to the top vertex, then remove the top vertex, resulting in a graph $G'(V',E')$. We have $|E'| = 2|V'| - 4$, which implies that G' is not rigid. The same argument holds when removing any edge not incident to the bottom vertex, then removing the bottom vertex. Thus, removing any edge in G destroys 2-

vertex-rigidity, and therefore G is weakly minimal 2-vertex-rigid. The same analysis holds for the graph in Figure 6(b) and all other graphs in this class. ■

Clearly, the existence of weakly minimal 2-vertex-rigid graphs make a complete characterization of 2-vertex-rigidity rather subtle and difficult. We conclude this section with a conjecture that the *X-replacement* and *degree 3 vertex addition* operations are sufficient to "grow" *all* weakly minimal 2-vertex-rigid graphs. We have already observed that these operations preserve weakly minimal 2-vertex-rigidity under certain conditions. For a complete characterization, we need to show that the reverse operations can always be applied to a weakly minimal 2-vertex-rigid graph.

Conjecture 1: Let $G(V,E)$ be a weakly minimal 2-vertex-rigid graph with at least 9 vertices. Then there exists either (a) a degree 4 vertex on which a reverse *X-replacement* operation can be performed to obtain a weakly minimal 2-vertex-rigid graph, or (b) there exists a degree three vertex on which a reverse *degree 3 vertex addition* can be performed to obtain a weakly minimal 2-vertex-rigid graph.

IV. STRONGLY MINIMAL 2-VERTEX-GLOBAL-RIGIDITY

In this section, we investigate the structure of graphs with the property that global rigidity is preserved when any single vertex and its incident edges are removed. Analogously, a graph is called *2-vertex-globally-rigid* if it remains globally rigid after removing any single vertex. Information architectures with this structure would allow self-localization in sensor networks to be performed even in the event of loss of any one. We provide a complete characterization of *strongly minimal 2-vertex-global-rigidity*.

We begin with the following result, which gives a lower bound on the number of edges necessary for 2-vertex-globally-rigidity.

Lemma 2: If $G(V,E)$ is a 2-vertex-globally-rigid graph on 5 or more vertices, then $|E| \geq 2|V|$ (excess three or more).

Proof: Suppose $G(V,E)$ is a 2-vertex-globally-rigid graph on 5 or more vertices with $|E| = 2|V| - 1$. The average degree in a graph on $|V|$ vertices with $|E| = 2|V| - 1$ is $4 - 2/|V|$. Thus, such a graph on 5 or more vertices has a vertex of degree at least 4. Removing such a vertex results in a graph $G'(V',E')$ with $|E'| = 2|V'| - 3$, which cannot be redundantly rigid and therefore cannot be globally rigid. This contradicts our original assumption that G was 2-vertex-globally-rigid. ■

The following two results completely characterize the structure of strongly minimal 2-vertex-globally-rigid graphs.

Theorem 6: Let $G(V,E)$ be a strongly minimal 2-vertex-globally-rigid on 5 or more vertices. Then we have the following:

- $|E| = 2|V|$.
- Every vertex in G has degree 4.

Proof: For the first condition, removing an edge results in a graph of excess two, which is not 2-vertex-globally-rigid by Lemma 2. Further, Figure 7 shows a 2-vertex-globally rigid graph where $|E| = 2|V|$. For the second condition, since G is globally rigid, it contains no vertex of degree less than 3. Suppose v is a vertex of degree 3. Remove a neighbour of v. In the resulting graph, v has degree 2 and this graph is not globally rigid. Since $|E| = 2|V|$, then the average vertex degree is 4. Since there are no vertices of degree 3 or fewer, then there cannot be any vertices of degree 5 or more. Thus, all vertices must have degree 4, which completes the proof. ■

Theorem 7: A graph $G(V,E)$ is strongly minimal 2-vertex-globally-rigid if and only if the following conditions hold

- $|E| = 2|V|$
- G is 4-vertex-connected
- G is redundantly strongly minimal 2-vertex-rigid (i.e. removing any edge results in a strongly minimal 2-vertex-rigid graph).

Proof: For sufficiency, suppose the conditions hold for a graph G. Note first that since G is 4-vertex-connected, a graph obtained by removing a vertex and its incident edges is 3-vertex-connected. Further, 4-vertex-connectivity implies that every vertex has degree at least 4, (see e.g. [21]), and since $|E| = 2|V|$ then every vertex has precisely degree 4. Now choose any vertex v in G and remove any edge incident to this vertex. The resulting graph is 2-vertex-rigid by the third condition. Remove another edge incident to v. Via the edge partition in Theorem 5, the resulting graph consists of v with degree 2 attached to a redundantly rigid graph. Now we can remove v and the resulting graph is redundantly rigid. By the second condition, it is also 3-vertex-connected and therefore is globally rigid. The argument holds for any vertex v in G, which proves that G is 2-vertex-globally-rigid, and thus the conditions are sufficient.

The 4-vertex-connectivity of G is obviously necessary because G minus any vertex must be 3-vertex-connected. Further, $|E| = 2|V|$ is necessary because if $|E| < 2|V|$ then there is a vertex with degree 3, which implies that G is not 4-vertex-connected. Now we need to prove the necessity of the final condition. To obtain a contradiction, suppose G is a 2-vertex-globally-rigid graph with an edge e that when removed does not result in a 2-vertex-rigid graph. Remove such an edge e and call the resulting graph G'. This implies that there exists a vertex v in G' that when removed results in a non-rigid graph G''. There are two cases: First, if e is incident to v, then effectively we have removed v from G to obtain a non-rigid graph G''. Thus, G is not 2-vertex-globally-rigid, contradicting our assumption. Second, if e is not incident to v, then if G is 2-vertex-globally-rigid, we should be able to reinsert e into G'' to obtain a globally rigid graph. However, it is impossible to add a single edge to a non-rigid graph to make it redundantly rigid. This again contradicts our assumption, which proves the necessity of the final condition and completes the proof. ■

Theorem 7 is clearly analogous to the characterization of global rigidity given by Theorem 3. An example of a 2-vertex-globally-rigid graph is given in Figure 7. This example is a C^2 graph, which can be obtained starting with a cycle on n vertices and connecting 2-hop neighbours. One could

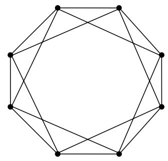

Fig. 7. Example of a strongly minimal 2-vertex-globally-rigid graph described in Theorem 7.

also obtain a different 2-vertex-globally-rigid graph by starting with a cycle on n vertices and connecting 3-hop neighbours. The smallest strongly minimal 2-vertex-globally-rigid graph is the complete graph on 5 vertices.

Operations to "grow" all strongly minimal 2-vertex-globally-rigid graphs and existence of weakly minimal 2-vertex-globally-rigid graphs remain open questions. In addition, note that generalizing Theorem 7 will require a complete characterization of 2-vertex-rigidity, again emphasizing the importance of resolving the difficulties discussed in the previous section.

V. CONCLUDING REMARKS

In summary, we have addressed the problem of agent loss by introducing redundancy into the information architecture such that agent loss does not destroy desirable properties. Specifically, we have investigate the structure of graphs with the property that rigidity or global rigidity is preserved after removing any single vertex, which we call *2-vertex-rigidity* or *2-vertex-global-rigidity*, respectively. Information architectures with such properties would allow critical tasks, such as formation shape maintenance or self-localization, to be performed even in the event of loss agent any single agent. We have reviewed a characterization from [13] of what we call *strongly minimal 2-vertex-rigidity* and further developed the class of what we call *weakly minimal 2-vertex-rigidity*, making significant strides toward a complete characterization. We have also presented for the first time a characterization of *strongly minimal 2-vertex-global-rigidity*.

The existence of weakly minimal 2-vertex-rigidity makes a complete characterization of 2-vertex-rigidity rather subtle and difficult, in comparison to rigidity. The existence of weakly minimal 2-vertex-global-rigidity remains open. However, we remain motivated to work toward complete characterizations because it will be fundamental for further theoretic and practical developments. For example, such characterizations will likely be instrumental in investigating *k-vertex-rigidity* and *k-vertex-global-rigidity* for $k > 2$, which would address robustness to loss of more than one agent. Moreover, complete characterizations would be helpful in developing efficient algorithms for determining whether a graph is *2-vertex-rigid* or *2-vertex-global-rigid*.

ACKNOWLEDGMENTS

T. H. Summers is a Fulbright Postgraduate Scholar supported by the Australian-American Fulbright Commission. C. Yu is supported by the Australian Research Council through an Australian Postdoctoral Fellowship under DP-0877562. B.D.O. Anderson is supported by DP-0877562 and National ICT Australia-NICTA. NICTA is funded by the Australian Government as represented by the Department of Broadband, Communications and the Digital Economy and the Australian Research Council through the ICT Centre of Excellence program.

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