Correction to "On Submodularity and Controllability in Complex Dynamical Networks"

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***Abstract*—We provide a correction to our paper "On Submodularity and Controllability in Complex Dynamical Networks", which appeared in Volume 3, Issue 1 of the IEEE Transactions on Control of Network Systems [4]. At the time of submission, and after our discovery of the error, we were informed of a related observation documented in [1].**

Our paper "On Submodularity and Controllability in Complex Dynamical Networks" published in Volume 3, Issue 1 of the IEEE Transactions on Control of Network Systems [4] contains an incorrect proof of Theorem 5. We give a counterexample[1] that invalidates the result.

Let A be a stable system dynamics matrix of a linear dynamical system and $V = \{b_1, ..., b_M\}$ be a set of possible columns that can be used to form or modify the system input matrix. For a given $S \subseteq V$, we form $B_S = [B_0 \quad b_s]$ given an existing input matrix B_0 and using the associated columns defined by $s \in S$, so that the state space representation is

$$\dot{x}(t) = Ax(t) + B_S u(t).$$

We assume that the pair (A, B_0) is controllable. We denote the controllability Gramian associated with S by

$$W_S = \int_0^\infty e^{A\tau} B_S B_S^T e^{A^T\tau} d\tau.$$

The paper contained the following theorem regarding submodularity of the trace of the inverse controllability Gramian with respect to actuator subsets:

Theorem 1. *Let $V = \{b_1, ..., b_M\}$ be a set of possible input matrix columns and W_S the controllability Gramian associated with $S \subseteq V$. The set function $f : 2^V \to \mathbf{R}$ defined as*

$$f(S) = -\mathbf{tr}(W_S^{-1})$$

is submodular and monotone increasing.

Unfortunately, further investigation revealed that the proof of this claim contains a subtle error. It effectively relies on a statement that for two positive definite matrices P and Q, $P^{-1} \succeq Q^{-1}$ implies that $P^{-2} \succeq Q^{-2}$. However, this is incorrect in general, since the partial ordering of positive semidefinite matrices is not necessarily preserved by squaring (or by any matrix power greater than one).

The following counterexample demonstrates that, unfortunately, the result is also incorrect, not just the proof. Consider the stable system dynamics matrix

$$A = \begin{bmatrix} -3.9 & -0.2 & 0.7 & -0.4 & 0 \\ -0.3 & -2.2 & 0.3 & 1.6 & -1.3 \\ 0.5 & 1 & -0.5 & 0.7 & -0.3 \\ 0 & 0.6 & -1.1 & 0.6 & 0.2 \\ 0 & 0.5 & 1.3 & 1.2 & -2.3 \end{bmatrix}$$

and set of possible input matrix columns $V = \{e_1, e_2, e_3, e_4, e_5\}$, where e_i is the standard basis vector with 1 in the ith entry and zeros elsewhere. Consider the actuator subsets $S_1 = \{e_2, e_3, e_5\}$ and $S_2 = \{e_3, e_4, e_5\}$, so that $S_1 \cup S_2 = \{e_2, e_3, e_4, e_5\}$ and $S_1 \cap S_2 = \{e_3, e_5\}$. We have to the nearest integer

$$\begin{aligned} \mathbf{tr}(W_{S_1}^{-1}) = 388, \quad \mathbf{tr}(W_{S_2}^{-1}) = 420 \\ \mathbf{tr}(W_{S_1 \cup S_2}^{-1}) = 226, \quad \mathbf{tr}(W_{S_1 \cap S_2}^{-1}) = 515, \end{aligned} \tag{1}$$

so that

$$\mathbf{tr}(W_{S_1}^{-1}) + \mathbf{tr}(W_{S_2}^{-1}) - [\mathbf{tr}(W_{S_1 \cup S_2}^{-1}) + \mathbf{tr}(W_{S_1 \cap S_2}^{-1})] = 67.$$

This violates the definition of submodularity (stated in Definition 2 of [4]), so the set function defined in Theorem 1 is not submodular.

All other statements in the paper are, to the best of our knowledge, correct, including the main results in Theorems 4, 6, and 7.

Finally, the same error affects analogous arguments in [2], in the context of graph rigidity, and in [3], in the context of network coherence. It would appear that the same counterexample applies to a related result stated in Proposition 2 of [5].

References

[1] A. Olshevsky. On (non)supermodularity of average control energy. *arXiv:1609.08706*, 2016.

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[3] T. Summers, I. Shames, J. Lygeros, and F. Dörfler. Topology design for optimal network coherence. In *European Control Conference*, pages 575–580. IEEE, 2015.

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[5] V. Tzoumas, M.A. Rahimian, G.J. Pappas, and A. Jadbabaie. Minimal actuator placement with bounds on control effort. *IEEE Transactions on Control of Network Systems*, 3(1):67–78, 2016.

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[1] Strictly speaking, we provide strong numerical evidence supporting incorrectness of the result that relies on accuracy of numerical computations and correctness of the source code of either MATLAB or NumPy. It is not too difficult to generate other numerical counterexamples, so that the evidence becomes overwhelming, though it is worth noting that an analytical counterexample can be constructed, as described in [1].