Correction to "Rigid Network Design Via Submodular Set Function Optimization"

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Abstract—**We provide a correction to our paper "Rigid Network Design Via Submodular Set Function Optimization", which appeared in Volume 2, Issue 3 of the IEEE Transactions on Network Science and Engineering [1].**

Our paper "Rigid Network Design Via Submodular Set Function Optimization" published in Volume 2, Issue 3 of the IEEE Transactions on Network Science and Engineering [1] contains an incorrect proof for part of Theorem 5. We give a counterexample[1] that invalidates the result.

Let $G = (\mathcal{V}, \mathcal{E})$ denote a network graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$, let $\mathbf{p} : \mathcal{V} \to \mathbb{R}^{2|\mathcal{V}|}$ denote a vertex coordinate map that specifies a position for each vertex in the plane, let $\mathcal{E}_c$ denote the edge set of the complete graph, and let $R_{((\mathcal{V},\mathcal{E}),\mathbf{p})}$ denote the rigidity matrix associated with the network $((\mathcal{V}, \mathcal{E}), \mathbf{p})$.

The paper contained the following statement regarding submodularity of the trace of the rigidity Gramian pseudoinverse with respect to edge subsets:

Theorem 1 (Incorrect). *Let $\mathcal{E} \subseteq \mathcal{E}_c$ and let $X_\mathcal{E} = R^\top_{((\mathcal{V},\mathcal{E}),\mathbf{p})} R_{((\mathcal{V},\mathcal{E}),\mathbf{p})}$ be the rigidity Gramian associated with $\mathcal{E}$. Moreover, assume that there exists an edge set $\mathcal{E}^\star$ such that $((\mathcal{V}, \mathcal{E}^\star), \mathbf{p})$ is minimally rigid, i.e. $\text{rank}(R_{((\mathcal{V},\mathcal{E}^\star),\mathbf{p})}) = 2|\mathcal{V}| - 3$, and $\mathcal{E}^\star \subseteq \mathcal{E}$. The following set function is submodular and monotone increasing*

- *$f(\mathcal{E}) = -\text{trace}(X^\dagger_\mathcal{E})$, where $X^\dagger_\mathcal{E}$ denotes the Moore-Penrose pseudoinverse of $X_\mathcal{E}$.*

Unfortunately, further investigation revealed that the proof of this claim contains a subtle error. It effectively relies on a statement that for two positive definite matrices P and Q, $P^{-1} \succeq Q^{-1}$ implies that $P^{-2} \succeq Q^{-2}$. However, this is incorrect in general, since the partial ordering of positive semidefinite matrices is not necessarily preserved by squaring (or by any matrix power greater than one).

The following counterexample demonstrates that, unfortunately, the result is also incorrect, not just the proof. Consider the rigid network on 5 nodes with vertex positions $\mathbf{p} = (4,3), (3,3), (1,1), (4,2), (1,5)$ and given minimally rigid edge set $\mathcal{E} = \{(1,2), (1,3), (1,4), (1,5), (2,3), (2,4), (2,5)\}$. Let $S_1 = \mathcal{E} \cup \{(3,4)\}$ and $S_2 = \mathcal{E} \cup \{(4,5)\}$, so that $S_1 \cap S_2 = \mathcal{E}$ and $S_1 \cup S_2 = \mathcal{E} \cup \{(3,4), (4,5)\}$.

We have to the nearest tenth of an integer

$$\begin{aligned} \mathbf{tr}(X^\dagger_{S_1}) = 13.6, \quad \mathbf{tr}(X^\dagger_{S_2}) = 33.2 \\ \mathbf{tr}(X^\dagger_{S_1 \cup S_2}) = 10.1, \quad \mathbf{tr}(X^\dagger_{S_1 \cap S_2}) = 34.0, \end{aligned} \tag{1}$$

so that

$$\mathbf{tr}(X^\dagger_{S_1}) + \mathbf{tr}(X^\dagger_{S_2}) - [\mathbf{tr}(X^\dagger_{S_1 \cup S_2}) + \mathbf{tr}(X^\dagger_{S_1 \cap S_2})] = 2.7.$$

This violates the definition of submodularity (stated in Definition 2 of [1]), so the set function defined in Theorem 2 is not submodular.

All other statements in the paper are, to the best of our knowledge, correct, including the main results in the rest of Theorem 5 and in Theorems 4 and 6. In particular, Theorem 5 should read

Theorem 2. *Let $\mathcal{E} \subseteq \mathcal{E}_c$ and let $X_\mathcal{E} = R^\top_{((\mathcal{V},\mathcal{E}),\mathbf{p})} R_{((\mathcal{V},\mathcal{E}),\mathbf{p})}$ be the rigidity Gramian associated with $\mathcal{E}$. Moreover, assume that there exists an edge set $\mathcal{E}^\star$ such that $((\mathcal{V}, \mathcal{E}^\star), \mathbf{p})$ is minimally rigid, i.e. $\text{rank}(R_{((\mathcal{V},\mathcal{E}^\star),\mathbf{p})}) = 2|\mathcal{V}| - 3$, and $\mathcal{E}^\star \subseteq \mathcal{E}$. The following set function is submodular and monotone increasing*

- *$f(\mathcal{E}) = \log\left(\prod_{i=1}^{\text{rank}(X_\mathcal{E})} \lambda_i(X_\mathcal{E})\right)$ where $\lambda_i(X_\mathcal{E})$ is the i-th eigenvalue of $X_\mathcal{E}$ and $\lambda_1(X_\mathcal{E}) \geq \cdots \geq \lambda_n(X_\mathcal{E})$.*

Moreover, the methodological developments and numerical experiments are unaffected.

The error originated from a similar argument made in [2] in the context of network controllability, and also affects a result in [3] in the context of network coherence.

References

[1] I. Shames and T.H. Summers. Rigid network design via submodular set function optimization. *IEEE Transactions on Network Science and Engineering*, 2(3):84–96, 2015.

[2] T.H. Summers, F. Cortesi, and J. Lygeros. On controllability and submodularity in complex dynamical networks. *IEEE Transactions on Control of Network Systems*, 3(1):91–101, 2016.

[3] T.H. Summers, I. Shames, J. Lygeros, and F. Dörfler. Topology design for optimal network coherence. In *European Control Conference*, pages 575–580. IEEE, 2015.

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[1]Strictly speaking, we provide strong numerical evidence supporting incorrectness of the result that relies on accuracy of numerical computations and correctness of the source code of either MATLAB or NumPy. It is not too difficult to generate other numerical counterexamples, so that the evidence becomes overwhelming