Identification of Linear Systems with Multiplicative Noise from Multiple Trajectory Data ⋆

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Abstract

We study identification of linear systems with multiplicative noise from multiple trajectory data. A least-squares algorithm, based on exploratory inputs, is proposed to simultaneously estimate the parameters of the nominal system and the covariance matrix of the multiplicative noise. The algorithm does not need prior knowledge of the noise or stability of the system, but requires mild conditions of inputs and relatively small length for each trajectory. Identifiability of the noise covariance matrix is studied, showing that there exists an equivalent class of matrices that generate the same second-moment dynamic of system states. It is demonstrated how to obtain the equivalent class based on estimates of the noise covariance. Asymptotic consistency of the algorithm is verified under sufficiently exciting inputs and system controllability conditions. Non-asymptotic estimation performance is also analyzed under the assumption that system states and noise are bounded, providing vanishing high-probability bounds as the number of trajectories grows to infinity. The results are illustrated by numerical simulations.

Key words: linear system identification, multiplicative noise, multiple trajectories, non-asymptotic results

1 Introduction

The study of stochastic systems with multiplicative noise, i.e., noise multiplying with system states and inputs, has a long history in control theory [1], but is re-emerging in the context of complex networked systems and learning-based control. In contrast with the additive noise setting, multiplicative noise has the ability to capture the dependence of noise on system states and control inputs. This situation occurs in modern control systems as diverse as robotics with distance-dependent sensor errors [2], networked systems with noisy communication channels [3, 4], modern power networks with high penetration of intermittent renewables [5], turbulent fluid flow [6], and neuronal brain networks [7]. Linear systems with multiplicative noise are particularly attractive as a stochastic modeling framework because they remain simple enough to admit closed-form expressions for stabilization [8] and optimal control [1,9,10]. Identification of linear systems with multiplicative noise needs to be investigated, as a preliminary step of solving these problems in practice.

The first issue to be addressed is that identification of linear systems with multiplicative noise requires to estimate not only the nominal system matrices, but also the noise covariance matrix. This stands in contrast to the additive noise case where the noise covariance matrix has no bearing on the control design and may be omitted during system identification.

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The second issue we address is to perform system identification based on multiple input-state trajectory data, rather than a single trajectory. Multiple trajectory data arises in two broad situations: 1) episodic tasks where a single system is reset to an initial state after a finite run time, as encountered in iterative learning control and reinforcement learning [11]; 2) collecting data from multiple identical systems in parallel, for example, physical experiments [12] and social processes [13]. For multiple trajectory data, the length of each trajectory may be small, but a large number of trajectories can be obtained, by virtue of repetition in the case of episodic tasks and parallel execution in that of multiple identical systems.

1.1 Related Work

For identification of a nominal linear system, recursive algorithms have been developed in the control literature, such as the recursive least-squares algorithm [14–16]. These can be utilized to identify linear systems with multiplicative noise provided that certain conditions for noise and system stability hold. Non-asymptotic performance analysis of identification methods can be found in [17–19]. It has once again attracted much attention from different domains and been investigated more extensively, because of recent development of random matrix theories, self-normalized martingales, and so on (see [20–22] and references therein).

For estimation of noise covariance, both recursive and batch methods have been proposed over the last few decades [23], but most of these methods focus on the additive noise case. In order to estimate multiplicative noise covariance, a maximum-likelihood approach is introduced in [24], and a Bayesian framework is utilized in [25, 26]. These methods, however, require prior assumptions on the noise distributions, whose incorrectness may worsen the performance of the algorithms. The papers [27, 28] study stochastic linear quadratic regulator (LQR) design for a special case of linear systems with multiplicative noise. It is assumed that the multiplicative noise is observed directly so that a concentration inequality can be obtained for the estimation of the noise covariance. The authors in [29] develop, concurrently and independently of the present work, finite-sample error bounds associated with simultaneously estimating the nomial system parameters and noise covariance matrix, by using single trajectory data, which is the most relevant work to ours. A self-normalizing (ellipsoidal) bound and a Euclidean (box) bound are provided for the least-squares estimation, but it is not sure whether the bounds converge to zero under the dynamic system setting.

There is a growing interest in system identification based on multiple trajectory data, along with their applications in data-driven control [20, 21], due to the powerful and convenient estimator schemes facilitated by resetting the system. This framework can be applied for both stable and unstable systems, because of the finite duration of each trajectory. The procedure of collecting multiple trajectories is utilized in [30, 31], to identify finite impulse response systems. In [20], a framework called coarse-ID control is introduced to solve the problem of LQR with unknown linear dynamics. The first step of this framework is to learn a coarse model of the unknown linear system, by observing multiple independent trajectories with finite length of the system. However, only the last input-state pairs of the trajectories are used in developing theoretical guarantees for the learning task. The performance of a least-squares algorithm, using all samples of every trajectory, is studied in [22], for estimating partially observed, possibly open-loop unstable, linear systems.

1.2 Contributions

In this paper we consider the identification of linear systems with multiplicative noise from multiple trajectory data. Our contributions are three-fold:

1. A least-squares estimation algorithm (Algorithm 1) is proposed to jointly estimate the nominal system matrices and multiplicative noise covariance from multiple trajectory data. The algorithm does not need prior knowledge of the noise or stability of the system, but requires mild conditions of inputs, relatively small length for each trajectory, and the assumption of independent and identically distributed (i.i.d.) noise with finite first and second moments.
2. Identifiability of the noise covariance matrix is investigated (Propositions 1 and 2). It is shown that there exists an equivalent class of covariance matrices that generate the same second-moment dynamic of system states. In addition, it is studied when such equivalent class has a unique element, meaning that the covariance matrix can be uniquely determined. An explicit expression of the equivalent class is provided so that the noise covariance can be recovered based on estimates given by the proposed algorithm.
3. Asymptotic consistency of the proposed algorithm is verified (Theorem 1), under sufficiently exciting inputs and system controllability conditions. Non-asymptotic estimation performance is also analyzed under the assumption

that the system is bounded, providing vanishing high-probability bounds as the number of trajectories grows to infinity (Theorems 2 and 3).

The differences between this paper and its conference version [32] are as follows. We study the identifiability of the noise covariance matrix in detail, demonstrating a framework to recover the equivalent class of the covariance matrix. Moreover, we give sharper bounds for the required length of each trajectory in Propositions 3 and 4. Finally, finite sample analysis of the proposed algorithm is provided.

1.3 Outline

The remainder of the paper is organized as follows: In Section 1.4 we provide notations used in the paper. We formulate the problem in Section 2. In Section 3 the algorithm is introduced and theoretical results are given. Numerical simulation results are presented in Section 4. In Section 5 we conclude the paper. Some proofs are postponed to Appendix.

1.4 Notation

We denote the n-dimensional Euclidean space by $\mathbb{R}^n$, and the set of $n \times m$ real matrices by $\mathbb{R}^{n\times m}$. Let $\mathbb{N}$ stand for the set of nonnegative integers, and $\mathbb{N}^+ = \mathbb{N} \setminus \{0\}$. Let $[k] := \{1, 2, \dots, k\}$, $k \in \mathbb{N}^+$. We use $\|\cdot\|$ to denote the Euclidean norm for vectors, and use $\|\cdot\|_F$ and $\|\cdot\|_2$ to denote the Frobenius and spectral norm for matrices. The probability of an event E is denoted by $\mathbb{P}\{E\}$, and the expectation of a random vector x is represented by $\mathbb{E}\{x\}$. An event happens almost surely (a.s.) means that it happens with probability one. Let $A \times B$ be the Cartesian product of sets A and B, i.e., $A \times B = \{(a, b) : a \in A, b \in B\}$. For two sequences of real numbers a_k and $b_k \neq 0$, $k \in \mathbb{N}^+$, we say $a_k = \mathcal{O}(b_k)$, if there exists a positive constant C such that $|a_k/b_k| \leq C$ for all $k \in \mathbb{N}^+$.

We use a_{ij} or $[A]_{ij}$ to represent the (i, j)-th entry of $A \in \mathbb{R}^{n\times m}$. Denote the n-dimensional all-one vector and all-zero vector by $\mathbf{1}_n$ and $\mathbf{0}_n$ respectively. The n-dimensional unit vector with i-th component being one is represented by $\mathbf{e}_i^n$. Denote the n-dimensional identity matrix by I_n. For two symmetric matrices $A, B \in \mathbb{R}^{n\times n}$, $A \succeq 0$ ($A \succ 0$) means that A is positive semidefinite (positive definite), and $A \succeq B$ ($A \succ B$) means $A - B \succeq 0$ ($A - B \succ 0$). For a matrix $A \in \mathbb{R}^{n\times n}$, $\rho(A)$ is used to represent the spectral radius of A. For a symmetric matrix $A \in \mathbb{R}^n$, denote its smallest and largest eigenvalue by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ respectively. A block diagonal matrix A with $A_1, \dots, A_k$ on its diagonal is denoted by $\mathrm{blockdiag}(A_1, \dots, A_k)$.

The Kronecker product of two matrices $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is represented by $A \otimes B$. The full vectorization of $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ is found by stacking the columns of A, i.e., $\mathrm{vec}(A) = (a_{11}\ a_{21}\ \cdots\ a_{m1}\ a_{12}\ a_{22}\ \cdots\ a_{mn})^\intercal$. The symmetric vectorization (sometimes called half-vectorization) of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is found by stacking the upper triangular part of the columns of A, i.e., by $\mathrm{svec}(A) = (a_{11}\ a_{12}\ a_{22}\ \cdots\ a_{1n}\ a_{2n}\ \cdots\ a_{nn})^\intercal$. The inverse operations of $\mathrm{vec}(\cdot)$ and $\mathrm{svec}(\cdot)$ are the full matricization $\mathrm{mat}_{p\times q}(x) := (\mathrm{vec}(I_q)^\intercal \otimes I_p)(I_q \otimes x)$ of a vector $x \in \mathbb{R}^{pq}$ and symmetric matricization $\mathrm{smat}_p(y)$ of a vector $y \in \mathbb{R}^{p(p+1)/2}$, respectively. Generalizing the vectorization and matricization operations to a block matrix

$$B = \begin{bmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ \vdots & \vdots & & \vdots \\ B_{m1} & B_{m2} & \cdots & B_{mn} \end{bmatrix} \in \mathbb{R}^{mp\times nq},$$

where $B_{ij} \in \mathbb{R}^{p\times q}$, we define the following matrix reshaping operator $F : \mathbb{R}^{mp\times nq} \to \mathbb{R}^{mn\times pq}$,

$$F(B, m, n, p, q) :=[\mathrm{vec}(B_{11})\ \ \mathrm{vec}(B_{21})\ \ \cdots\ \ \mathrm{vec}(B_{m1})\ \ \cdots\ \ \mathrm{vec}(B_{12})\ \ \mathrm{vec}(B_{22})\ \ \cdots\ \ \mathrm{vec}(B_{mn})]^\intercal.$$

Then we have that $F(A \otimes A, m, n, m, n) = \mathrm{vec}(A)\,\mathrm{vec}(A)^\intercal$ for $A \in \mathbb{R}^{m\times n}$, which demonstrates the correspondence between the entries of $A \otimes A$ and those of $\mathrm{vec}(A)\,\mathrm{vec}(A)^\intercal$. Note when $p = q = 1$, $F(\cdot)$ degenerates to $\mathrm{vec}(\cdot)$. We also define the inverse reshaping operator $G : \mathbb{R}^{mn\times pq} \to \mathbb{R}^{mp\times nq}$ as

$$G(B, m, n, p, q) := \begin{bmatrix} \mathrm{mat}_{p\times q}(B_1) & \cdots & \mathrm{mat}_{p\times q}(B_{(n-1)m+1}) \\ \mathrm{mat}_{p\times q}(B_2) & \cdots & \mathrm{mat}_{p\times q}(B_{(n-1)m+2}) \\ \vdots & & \vdots \\ \mathrm{mat}_{p\times q}(B_m) & \cdots & \mathrm{mat}_{p\times q}(B_{mn}) \end{bmatrix},$$

where $B \in \mathbb{R}^{mn\times pq}$, $B_i^\intercal$ is the i-th row of B. Thus F and G are inverses of each other in the sense that

$$F(G(A, m, n, p, q), m, n, p, q) = A,$$
$$G(F(B, m, n, p, q), m, n, p, q) = B,$$

for any $A \in \mathbb{R}^{mn\times pq}$ and $B \in \mathbb{R}^{mp\times nq}$. In this way, we have $G(\mathrm{vec}(A)\,\mathrm{vec}(A)^\intercal, m, n, m, n) = A \otimes A$ for $A \in \mathbb{R}^{m\times n}$. Note that both F and G are linear, i.e., $F(A+B, m, n, p, q) = F(A, m, n, p, q) + F(B, m, n, p, q)$ for $A, B \in \mathbb{R}^{mp\times nq}$, and $G(A+B, m, n, p, q) = G(A, m, n, p, q) + G(B, m, n, p, q)$ for $A, B \in \mathbb{R}^{mn\times pq}$.

2 Problem Formulation

We consider linear systems with multiplicative noise

$$x_{t+1} = (A + \bar{A}_t)x_t + (B + \bar{B}_t)u_t,\ t \in \mathbb{N}, \tag{1}$$

where $x_t \in \mathbb{R}^n$ is the system state, and $u_t \in \mathbb{R}^m$ is the control input, $m \le n$. The system is described by nominal dynamic matrix $A \in \mathbb{R}^{n\times n}$ and nominal input matrix $B \in \mathbb{R}^{n\times m}$, and incorporates multiplicative noise terms modeled by i.i.d. and mutually independent random matrices $\bar{A}_t$ and $\bar{B}_t$, which have zero mean and covariance matrices $\Sigma_A := \mathbb{E}\{\mathrm{vec}(\bar{A}_t)\,\mathrm{vec}(\bar{A}_t)^T\} \in \mathbb{R}^{n^2\times n^2}$ and $\Sigma_B := \mathbb{E}\{\mathrm{vec}(\bar{B}_t)\,\mathrm{vec}(\bar{B}_t)^T\} \in \mathbb{R}^{nm\times nm}$. The multiplicative noise is independent of the inputs. Note that if $\bar{A}_t$ has non-zero mean $\bar{A}$, then we can consider a system with nominal matrix $[A + \bar{A}\ B]$, as well as noise terms $\bar{A}_t - \bar{A}$ and $\bar{B}_t$, which satisfies the above zero-mean assumption. This also holds for the case of $\bar{B}_t$ with non-zero mean. The term multiplicative noise refers to that noise, $\bar{A}_t$ and $\bar{B}_t$, enters the system as multipliers of x_t and u_t, rather than as additions. The independence of $\bar{A}_t$ and $\bar{B}_t$ is assumed for simplicity, and under this assumption the covariance matrix of the entire multiplicative noise is a block diagonal matrix $\mathbb{E}\{\mathrm{vec}([\bar{A}_t\ \bar{B}_t])\,\mathrm{vec}([\bar{A}_t\ \bar{B}_t])^\intercal\} = \mathrm{blockdiag}(\Sigma_A, \Sigma_B)$. Throughout the paper, we will use $(\Sigma_A, \Sigma_B) \in \mathbb{R}^{n^2\times n^2} \times \mathbb{R}^{nm\times nm}$ to represent this matrix. If $\bar{A}_t$ and $\bar{B}_t$ are dependent, there is an extra but amenable term on their correlations, $\mathbb{E}\{\mathrm{vec}(\bar{A}_t)\,\mathrm{vec}(\bar{B}_t)^\intercal\}$.

As an example of System (1), consider the following system studied in the optimal control literature [8, 10].

$$x_{t+1} = \left(A + \sum_{i=1}^{r} A_i p_{i,t}\right)x_t + \left(B + \sum_{j=1}^{s} B_j q_{j,t}\right)u_t, \tag{2}$$

where $\{p_{i,t}\}$ and $\{q_{i,t}\}$ are mutually independent scalar random variables, with $\mathbb{E}\{p_{i,t}\} = \mathbb{E}\{q_{j,t}\} = 0$, $\mathbb{E}\{p_{i,t}^2\} = \sigma_i^2$, and $\mathbb{E}\{q_{j,t}^2\} = \delta_j^2$, $\forall i \in [r], j \in [s], t \in \mathbb{N}$. It can be seen that $\bar{A}_t = \sum_{i=1}^r A_i p_{i,t}$ and $\bar{B}_t = \sum_{j=1}^s B_j q_{j,t}$, where σ_i and δ_j are the eigenvalues of Σ_A and Σ_B, and A_i and B_j are the reshaped eigenvectors of Σ_A and Σ_B. These parameters are necessary for optimal controller design [10]. However, for new systems with unknown parameters, the key problem is to identify them in the first place. Another example of System (1) is interconnected systems, where the nominal part captures relations among different subsystems, and multiplicative noise characterizes randomly varying topologies [33].

Suppose that multiple state-input trajectories $\{(x_t^{(k)}, u_t^{(k)}), 0 \le t \le \ell, k \in \mathbb{N}^+\}$ are available, where $\{(x_t^{(k)}, u_t^{(k)}), 0 \le t \le \ell\}$ is the k-th trajectory, and ℓ is the rollout length (index of the final time-step) for every trajectory. The problem considered in this paper is as follows.

Problem. Given multiple trajectory data $\{(x_t^{(k)}, u_t^{(k)}), 0 \le t \le \ell, k \in \mathbb{N}^+\}$, estimate the nominal system matrix $[A\ B]$ and the noise covariance matrix (Σ_A, Σ_B).

3 Least-Squares Algorithm Based on Multiple Trajectory Data

In this section, identifiability of the noise covariance matrix is studied in Section 3.1, paving the way to algorithm design. Consistency of the algorithm is given by Theorem 1 in Section 3.2. Finally, sample complexity of the algorithm is studied in Section 3.3, and qualitative results are provided in Theorems 2 and 3.

3.1 Moment Dynamics and Algorithm Design

In this section, we propose a least-squares algorithm to estimate system parameters from multiple trajectory data. We assume that the sampled trajectory data are collected independently, and refer to each trajectory sample as a *rollout*. First of all, we study the effect of multiplicative noise by investigating the moment dynamics of system states. This relates to an identifiability issue of the noise covariance matrix, which needs to be clarified before stating the algorithm.

Taking the expectation of both sides of System (1) and letting $\mu_t := \mathbb{E}\{x_t\}$ and $\nu_t := \mathbb{E}\{u_t\}$, we obtain the first-moment dynamic of system states, i.e., the dynamic of $\mathbb{E}\{x_t\}$, as follows,

$$\mu_{t+1} = A\mu_t + B\nu_t,\ t \in \mathbb{N}. \tag{3}$$

Likewise, denote the vectorization of the second-moment matrices of state, state-input, and input at time t by $X_t := \mathrm{vec}(\mathbb{E}\{x_t x_t^\intercal\})$, $W_t := \mathrm{vec}(\mathbb{E}\{x_t u_t^\intercal\})$, $W_t' := \mathrm{vec}(\mathbb{E}\{u_t x_t^\intercal\})$, and $U_t := \mathrm{vec}(\mathbb{E}\{u_t u_t^\intercal\})$. Note that the second moment matrix $\mathbb{E}\{xy^\intercal\}$ for two random vectors x and y is different from the second central moment (covariance) matrix $\mathbb{E}\{(x - \mathbb{E}\{x\})(y - \mathbb{E}\{y\})^\intercal\} = \mathbb{E}\{xy^\intercal\} - \mathbb{E}\{x\}\mathbb{E}\{y\}^\intercal$, whenever both x and y have nonzero mean.

From the independence of $\bar{A}_t$ and $\bar{B}_t$, as well as vectorization, the second-moment dynamic of system states is

$$\begin{aligned} X_{t+1} &= (A\otimes A)X_t + (B\otimes A)W_t + (A\otimes B)W_t' + (B\otimes B)U_t + \mathbb{E}\left\{(\bar{A}_t\otimes\bar{A}_t)\,\mathrm{vec}(x_t x_t^\intercal)\right\} + \mathbb{E}\left\{(\bar{B}_t\otimes\bar{B}_t)\,\mathrm{vec}(u_t u_t^\intercal)\right\} \\ &= (A\otimes A + \Sigma_A')X_t + (B\otimes B + \Sigma_B')U_t + (B\otimes A)W_t + (A\otimes B)W_t',\ t\in\mathbb{N}, \end{aligned} \tag{4}$$

where $\Sigma_A' = \mathbb{E}\{\bar{A}_t \otimes \bar{A}_t\} \in \mathbb{R}^{n^2\times n^2}$ and $\Sigma_B' = \mathbb{E}\{\bar{B}_t\otimes\bar{B}_t\} \in \mathbb{R}^{n^2\times m^2}$. The relation between (Σ_A, Σ_B) and (Σ_A', Σ_B') can be illustrated by $F(\Sigma_A', n, n, n, n) = \Sigma_A$ and $F(\Sigma_B', n, m, n, m) = \Sigma_B$, where the reshaping operator $F(\cdot)$ is defined in Section 1.4. As said earlier, if $\bar{A}_t$ and $\bar{B}_t$ are dependent, then there are two extra terms, $\mathbb{E}\{\bar{B}_t\otimes\bar{A}_t\}W_t$ and $\mathbb{E}\{\bar{A}_t\otimes\bar{B}_t\}W_t'$, in (4).

Before giving an estimation algorithm, it is necessary to discuss an intrinsic identifiability issue arising in second-moment dynamic (4). Since $\mathbb{E}\{x_t x_t^\intercal\}$ is symmetric, X_t has $n(n-1)/2$ pairs of identical entries corresponding to the off-diagonal entries of $\mathbb{E}\{x_t x_t^\intercal\}$, i.e., $\mathbb{E}\{x_{t,i}x_{t,j}\} = \mathbb{E}\{x_{t,j}x_{t,i}\}$ for all $i, j \in [n]$. In order to remove the redundant terms in (4), we refer to binary row- and column-selection matrices, also called elimination and duplication matrices [34].

To begin, notice that the redundant entries of X_t are associated with the index set $\{(j-1)n + i : i, j \in [n], i < j\}$. Define matrix $T_1 \in \mathbb{R}^{n^2\times n^2}$ by replacing the $[(j-1)n+i]$-th row of I_{n^2} by $(\mathbf{e}^{n^2}_{(i-1)n+j})^\intercal$ for all $i, j \in [n]$ with $i < j$. Then, by noticing that $\mathbb{E}\{x_{t,i}x_{t,j}\}$ is the $[(j-1)n+i]$-th entry of X_t, it follows that X_t is invariant under T_1, i.e., $X_t = T_1 X_t$. Furthermore, we define a binary elimination matrix P_1 that picks out only the unique entries of X_t as well as a complementary binary duplication matrix Q_1 which in turn reconstructs X_t from the unique representation, by repeating the redundant entries in the proper order. These matrices are defined explicitly as $P_1 \in \mathbb{R}^{[n(n+1)/2]\times n^2}$ by removing the $[(j-1)n+i]$-th row of I_{n^2}, $i, j \in [n]$ with $i < j$, and $Q_1 \in \mathbb{R}^{n^2\times[n(n+1)/2]}$ by removing the $[(j-1)n+i]$-th column of T_1, $i, j \in [n]$ with $i < j$. Then we may freely convert between the full vectorization (with redundant entries) X_t and the symmetric vectorization (without redundant entries) $\tilde{X}_t := \mathrm{svec}(X_t)$ by employing the linear transformations defined by the matrices P_1 and Q_1:

$$\tilde{X}_t = P_1 X_t, \qquad X_t = Q_1 \tilde{X}_t.$$

The same arguments apply to the second moment of input, U_t. That is, U_t has $m(m-1)/2$ pairs of identical entries corresponding to the off-diagonal entries of $\mathbb{E}\{u_t u_t^\intercal\}$, so we define $T_2 \in \mathbb{R}^{m^2\times m^2}$, $P_2 \in \mathbb{R}^{[m(m+1)/2]\times m^2}$, and $Q_2 \in \mathbb{R}^{m^2\times[m(m+1)/2]}$ by replacing n by m in the definitions of T_1, P_1, and Q_1, respectively.

By applying the symmetric vectorization transformations $\tilde{X}_t = P_1 X_t$ and $\tilde{U}_t = P_2 U_t$, the unique entries of the second-moment dynamic (4) can be characterized as

$$\begin{aligned}
\tilde{X}_{t+1} &= P_1 X_{t+1} \\
&= P_1(A \otimes A + \Sigma'_A)X_t + P_1(B \otimes B + \Sigma'_B)U_t + P_1(B \otimes A)W_t + P_1(A \otimes B)W'_t \\
&= P_1(A \otimes A + \Sigma'_A)Q_1 P_1 X_t + P_1(B \otimes B + \Sigma'_B)Q_2 P_2 U_t + P_1(B \otimes A)W_t + P_1(A \otimes B)W'_t \\
&:= (\tilde{A} + \tilde{\Sigma}'_A)\tilde{X}_t + (\tilde{B} + \tilde{\Sigma}'_B)\tilde{U}_t + K_{BA}W_t + K_{AB}W'_t, \qquad (5)
\end{aligned}$$

where the penultimate equation follows from $T_1 = Q_1 P_1$ and $T_2 = Q_2 P_2$. In the last equation we introduce these notations:

$$\begin{aligned}
\tilde{A} &:= P_1(A \otimes A)Q_1 \in \mathbb{R}^{[n(n+1)/2] \times [n(n+1)/2]}, \\
\tilde{\Sigma}'_A &:= P_1 \Sigma'_A Q_1 \in \mathbb{R}^{[n(n+1)/2] \times [n(n+1)/2]}, \\
\tilde{B} &:= P_1(B \otimes B)Q_2 \in \mathbb{R}^{[n(n+1)/2] \times [m(m+1)/2]}, \\
\tilde{\Sigma}'_B &:= P_1 \Sigma'_B Q_2 \in \mathbb{R}^{[n(n+1)/2] \times [m(m+1)/2]}, \\
K_{BA} &:= P_1(B \otimes A), \ K_{AB} := P_1(A \otimes B).
\end{aligned}$$

Note that $\tilde{X}_t$ and $\tilde{U}_t$ have no redundant entries but are able to capture the second-moment dynamic of system states.

By the definition of Kronecker product, Σ'_A and Σ'_B have the structures shown in (6) respectively.

$$\begin{array}{c} \\ \\ (i-1)n+j \\ \\ (j-1)n+i \\ \\ \end{array}
\begin{array}{c}
\begin{array}{cc} (k-1)n+l & (l-1)n+k \end{array} \\
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & \mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\} & \cdots & \mathbb{E}\{[\bar{A}_t]_{il}[\bar{A}_t]_{jk}\} & \cdots \\
 & \vdots & & \vdots & \\
\cdots & \mathbb{E}\{[\bar{A}_t]_{jk}[\bar{A}_t]_{il}\} & \cdots & \mathbb{E}\{[\bar{A}_t]_{jl}[\bar{A}_t]_{ik}\} & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{array}
\begin{array}{c}
\begin{array}{cc} (p-1)m+q & (q-1)m+p \end{array} \\
\begin{bmatrix}
 & \vdots & & \vdots & \\
\cdots & \mathbb{E}\{[\bar{B}_t]_{ip}[\bar{B}_t]_{jq}\} & \cdots & \mathbb{E}\{[\bar{B}_t]_{iq}[\bar{B}_t]_{jp}\} & \cdots \\
 & \vdots & & \vdots & \\
\cdots & \mathbb{E}\{[\bar{B}_t]_{jp}[\bar{B}_t]_{iq}\} & \cdots & \mathbb{E}\{[\bar{B}_t]_{jq}[\bar{B}_t]_{ip}\} & \cdots \\
 & \vdots & & \vdots &
\end{bmatrix}
\end{array}, \quad (6)$$

where $i, j, k, l \in [n]$, $p, q \in [m]$, and $[\bar{A}_t]_{ij}$ ($[\bar{B}_t]_{ip}$) is the (i,j)-th entry of $\bar{A}_t$ ((i,p)-th entry of $\bar{B}_t$). If $i = j$ ($k = l$), the corresponding two rows (two columns) coincide. The proposition below demonstrates the entries of $\tilde{\Sigma}'_A$ and $\tilde{\Sigma}'_B$, and their correspondence with those of Σ'_A and Σ'_B.

Proposition 1 *Denote the (i,j)-th entry of $\tilde{\Sigma}'_A$ by $[\tilde{\Sigma}'_A]_{ij}$, then it holds that*

$$\begin{aligned}
&[\tilde{\Sigma}'_A]_{(i-1)(n-i/2)+i,(k-1)(n-k/2)+k} = \mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{ik}\}, \ i, k \in [n], \\
&[\tilde{\Sigma}'_A]_{(i-1)(n-i/2)+i,(k-1)(n-k/2)+l} = 2\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{il}\}, \ k < l, i, k, l \in [n], \\
&[\tilde{\Sigma}'_A]_{(i-1)(n-i/2)+j,(k-1)(n-k/2)+k} = \mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jk}\}, \ i < j, \ i, j, k \in [n], \\
&[\tilde{\Sigma}'_A]_{(i-1)(n-i/2)+j,(k-1)(n-k/2)+l} = \mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\} + \mathbb{E}\{[\bar{A}_t]_{il}[\bar{A}_t]_{jk}\}, \ i < j, \ k < l, \ i, j, k, l \in [n].
\end{aligned}$$

Denote the (i,j)-th entry of $\tilde{\Sigma}'_B$ by $[\tilde{\Sigma}'_B]_{ij}$, then it holds that

$$\begin{aligned}
&[\tilde{\Sigma}'_B]_{(i-1)(n-i/2)+i,(p-1)(m-p/2)+p} = \mathbb{E}\{[\bar{B}_t]_{ip}[\bar{B}_t]_{ip}\}, \ i \in [n], \ p \in [m], \\
&[\tilde{\Sigma}'_B]_{(i-1)(n-i/2)+i,(p-1)(m-p/2)+q} = 2\mathbb{E}\{[\bar{B}_t]_{ip}[\bar{B}_t]_{iq}\}, \ i \in [n], \ p < q, \ p, q \in [m], \\
&[\tilde{\Sigma}'_B]_{(i-1)(n-i/2)+j,(p-1)(m-p/2)+p} = \mathbb{E}\{[\bar{B}_t]_{ip}[\bar{B}_t]_{jp}\}, \ i < j, \ i, j \in [n], \ p \in [m], \\
&[\tilde{\Sigma}'_B]_{(i-1)(n-i/2)+j,(p-1)(m-p/2)+q} = \mathbb{E}\{[\bar{B}_t]_{ip}[\bar{B}_t]_{jq}\} + \mathbb{E}\{[\bar{B}_t]_{iq}[\bar{B}_t]_{jp}\}, \ i < j, \ p < q, \ i, j \in [n], \ p, q \in [m].
\end{aligned}$$

PROOF. By observing the definitions of P_i and Q_i, $i = 1, 2$, and the structures of Σ'_A and Σ'_B shown in (6), we can get the expressions of the entries of $\tilde{\Sigma}'_A$ and $\tilde{\Sigma}'_B$ as in the proposition. To determine their positions, note from the definition of P_1 that all of the $[(j-1)n+i]$-th rows of Σ'_A are removed during the transformation $P_1\Sigma'_A$, where $j > i$, $i,j \in [n]$. This means that the following rows above the $[(i-1)n+j]$-th row of Σ'_A, $i \leq j$, $i,j \in [n]$, are removed: $(i-1)n+1, \ldots, (i-1)n+i-1, (i-2)n+1, \ldots, (i-2)n+i-2, \ldots, n+1$, whose total number is $i(i-1)/2$. Thus, the $[(i-1)n+j]$-th rows of Σ'_A becomes the $[(i-1)n+j-i(i-1)/2]$-th row of $\tilde{\Sigma}'_A$, i.e., the $[(i-1)(n-i/2)+j]$-th row, where $i \leq j$, $i,j \in [n]$. Applying the same argument to the columns of Σ'_A and to Σ'_B, we obtain the correspondence given in the proposition.

Remark 1 *The above discussion indicates that X_t is in fact determined by $[A\ B]$ and $[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]$. It also shows that there exist a set of equivalent covariance matrices in the sense that they generate the same second-moment dynamic of system states, given nominal part $[A\ B]$. This is because the dynamic of $X_t = Q_1\tilde{X}_t$ only depends on $[A\ B]$ and $[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]$, and is invariant with respect to (Σ'_1, Σ'_2) satisfying $P_1\Sigma'_1Q_1 = \tilde{\Sigma}'_A$ and $P_2\Sigma'_2Q_2 = \tilde{\Sigma}'_B$.*

From an entry-wise point of view, $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\}$ and $\mathbb{E}\{[\bar{A}_t]_{il}[\bar{A}_t]_{jk}\}$, $i \neq j$ and $k \neq l$, have a coupled effect on the second-moment dynamic of system states. One may only estimate the sum of these two entries out of X_t, rather than their exact values. This is because we do not observe realizations of $\bar{A}_t$ and $\bar{B}_t$ directly, but indirectly through their effect on system states. Fortunately, some entries of Σ'_A and Σ'_B are identifiable, e.g., $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{ik}\}$, the variance of $[\bar{A}_t]_{ik}$, and $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jk}\}$, the covariance between entries in the same column. Similar issues also appear, when estimating covariance matrices, in topics such as Kalman filtering [35, 36]. Critically, since these identifiable quantities uniquely generate the second-moment dynamic of system states, it suffices to estimate $\tilde{\Sigma}'_A$ and $\tilde{\Sigma}'_B$ for the purposes of linear quadratic optimal control. This can be verified by expanding the Bellman equation; we omit the details here to keep the paper concise.

Given (Σ_A, Σ_B) with $\Sigma_A \succeq 0$ and $\Sigma_B \succeq 0$ (then $\tilde{\Sigma}'_A = P_1\Sigma'_AQ_1$ and $\tilde{\Sigma}'_B = P_2\Sigma'_BQ_2$), the set of equivalent matrices mentioned in Remark 1 can be written explicitly as follows, where positive semidefinite conditions are imposed because Σ_A and Σ_B are covariance matrices,

$$
\begin{aligned}
S^*(\tilde{\Sigma}'_A) &:= \left\{\Sigma_A(\alpha) \in \mathbb{R}^{n^2\times n^2} : \Sigma_A(\alpha) \succeq 0, \alpha \in \mathbb{R}^{n^2(n-1)^2/4}\right\},\\
S^*(\tilde{\Sigma}'_B) &:= \left\{\Sigma_B(\beta) \in \mathbb{R}^{nm\times nm} : \Sigma_B(\beta) \succeq 0, \beta \in \mathbb{R}^{nm(n-1)(m-1)/4}\right\},\\
S^*_\Sigma &:= S^*(\tilde{\Sigma}'_A) \times S^*(\tilde{\Sigma}'_B),
\end{aligned} \tag{7}
$$

with $\Sigma_A(\alpha) := F(Q_1\tilde{\Sigma}'_AQ_1^\intercal D_n + E_\alpha, n, n, n, n)$ and $\Sigma_B(\beta) := F(Q_1\tilde{\Sigma}'_BQ_2^\intercal D_m + E_\beta, n, m, n, m)$. Here

$$
\begin{aligned}
E_\alpha &= \sum_{\substack{i,j,k,l\in[n]\\ i<j,k<l}} \left[\alpha_{ij,kl}\left(\mathbf{e}^{n^2}_{(i-1)n+j} - \mathbf{e}^{n^2}_{(j-1)n+i}\right)\left(\mathbf{e}^{n^2}_{(k-1)n+l} - \mathbf{e}^{n^2}_{(l-1)n+k}\right)^\intercal\right],\\
E_\beta &= \sum_{\substack{i,j\in[n],i<j\\ p,q\in[m],p<q}} \left[\beta_{ij,pq}\left(\mathbf{e}^{n^2}_{(i-1)n+j} - \mathbf{e}^{n^2}_{(j-1)n+i}\right)\left(\mathbf{e}^{m^2}_{(p-1)m+q} - \mathbf{e}^{m^2}_{(q-1)m+p}\right)^\intercal\right],
\end{aligned}
$$

with $\alpha = [\alpha_{ij,kl}] \in \mathbb{R}^{n^2(n-1)^2/4}$, $\beta = [\beta_{ij,pq}] \in \mathbb{R}^{nm(n-1)(m-1)/4}$, $i,j,k,l \in [n]$, $p,q \in [m]$, $i<j$, $k<l$, $p<q$, Q_1 and Q_2 are given before (5), D_n is an n^2-dimensional diagonal matrix with $[(i-1)n+i]$-th diagonal entry being 1 and the rest being 1/2, $i \in [n]$, and D_m is an m^2-dimensional diagonal matrix with $[(p-1)m+p]$-th diagonal entry being 1 and the rest being 1/2, $p \in [m]$. Note that S^*_Σ is given by two inequalities which depend on α and β respectively. In fact, these two are linear matrix inequalities [8], since the reshaping operator F is linear. Obviously S^*_Σ is not empty, because (Σ_A, Σ_B) is one of its elements. The following examples provide an intuitive idea of the above discussion.

Example 1 *Consider System (1) with $n = 2$ and $m = 1$, where $X_t = [\mathbb{E}\{x_{t,1}x_{t,1}\}\ \mathbb{E}\{x_{t,2}x_{t,1}\}\ \mathbb{E}\{x_{t,1}x_{t,2}\}\ \mathbb{E}\{x_{t,2}x_{t,2}\}]^T$. So $\mathbb{E}\{X_{t,2}X_{t,1}\}$ and $\mathbb{E}\{X_{t,1}X_{t,2}\}$ are identical and have the same dynamic due to (4). Under this situation,*

$$
\tilde{X}_t = [\mathbb{E}\{x_{t,1}x_{t,1}\}\ \ \mathbb{E}\{x_{t,2}x_{t,1}\}\ \ \mathbb{E}\{x_{t,2}x_{t,2}\}]^T,
$$

$$P_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ Q_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ T_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \ P_2 = Q_2 = T_2 = 1.$$

According to the above simplification, from

$$\Sigma'_A = \begin{bmatrix} \sigma_{a,11,11} & \sigma_{a,11,12} & \sigma_{a,12,11} & \sigma_{a,12,12} \\ \sigma_{a,11,21} & \sigma_{a,11,22} & \sigma_{a,12,21} & \sigma_{a,12,22} \\ \sigma_{a,21,11} & \sigma_{a,21,12} & \sigma_{a,22,11} & \sigma_{a,22,12} \\ \sigma_{a,21,21} & \sigma_{a,21,22} & \sigma_{a,22,21} & \sigma_{a,22,22} \end{bmatrix}, \ \Sigma'_B = \begin{bmatrix} \sigma_{b,11} & \sigma_{b,12} & \sigma_{b,21} & \sigma_{b,22} \end{bmatrix}^T,$$

we have

$$\tilde{\Sigma}'_A = \begin{bmatrix} \sigma_{a,11,11} & 2\sigma_{a,11,12} & \sigma_{a,12,12} \\ \sigma_{a,11,21} & \sigma_{a,11,22} + \sigma_{a,12,21} & \sigma_{a,12,22} \\ \sigma_{a,21,21} & 2\sigma_{a,21,22} & \sigma_{a,22,22} \end{bmatrix}, \ \tilde{\Sigma}'_B = \begin{bmatrix} \sigma_{b,11} & \sigma_{b,12} & \sigma_{b,22} \end{bmatrix}^T,$$

where $\sigma_{a,ij,kl} = \mathbb{E}\{[\bar{A}_t]_{ij}[\bar{A}_t]_{kl}\}$, $\sigma_{b,ij} = \mathbb{E}\{[\bar{B}_t]_i, [\bar{B}_t]_j\}$, *and*

$$\tilde{A} = \begin{bmatrix} a_{11}a_{11} & a_{11}a_{12} + a_{12}a_{11} & a_{12}a_{12} \\ a_{11}a_{21} & a_{11}a_{22} + a_{12}a_{21} & a_{12}a_{22} \\ a_{21}a_{21} & a_{21}a_{22} + a_{22}a_{21} & a_{22}a_{22} \end{bmatrix}, \ \tilde{B} = \begin{bmatrix} b_1b_1 & b_1b_2 & b_2b_2 \end{bmatrix}^T,$$

$$K_{AB} = \begin{bmatrix} a_{11}b_1 & a_{12}b_1 \\ a_{21}b_1 & a_{22}b_2 \\ a_{21}b_2 & a_{22}b_2 \end{bmatrix}, \ K_{BA} = \begin{bmatrix} a_{11}b_1 & a_{12}b_1 \\ a_{11}b_2 & a_{12}b_2 \\ a_{21}b_2 & a_{22}b_2 \end{bmatrix}.$$

In this example, Σ_B *is unique, but based on* (7) *the covariance matrix equivalent to* Σ_A *is given by*

$$\Sigma_A(\alpha) = \begin{bmatrix} \sigma_{a,11,11} & \sigma_{a,11,21} & \sigma_{a,11,12} & \sigma_{a,11,22} + \alpha \\ \sigma_{a,21,11} & \sigma_{a,21,21} & \sigma_{a,21,12} - \alpha & \sigma_{a,21,22} \\ \sigma_{a,12,11} & \sigma_{a,12,21} - \alpha & \sigma_{a,12,12} & \sigma_{a,12,22} \\ \sigma_{a,22,11} + \alpha & \sigma_{a,22,21} & \sigma_{a,22,12} & \sigma_{a,22,22} \end{bmatrix},$$

where $\alpha \in \mathbb{R}$ *such that* $\Sigma_A(\alpha) \succeq 0$.

Example 2 *Consider System* (2) *with* $\bar{A}_t = \sum_{i=1}^r A_i p_{i,t}$, $\bar{B}_t = \sum_{j=1}^s B_j q_{j,t}$. *Hence,*

$$\Sigma_A = \sum_{i=1}^r \mathbb{E}\{p_{i,t}^2\} \operatorname{vec}(A_i) \operatorname{vec}(A_i)^\intercal, \ \Sigma'_A = \sum_{i=1}^r \mathbb{E}\{p_{i,t}^2\} A_i \otimes A_i, \ \tilde{\Sigma}'_A = \sum_{i=1}^r \mathbb{E}\{p_{i,t}^2\} P_1 (A_i \otimes A_i) Q_1,$$
$$\Sigma_B = \sum_{j=1}^s \mathbb{E}\{q_{j,t}^2\} \operatorname{vec}(B_j) \operatorname{vec}(B_j)^\intercal, \ \Sigma'_B = \sum_{j=1}^s \mathbb{E}\{q_{j,t}^2\} B_j \otimes B_j, \ \tilde{\Sigma}'_B = \sum_{j=1}^s \mathbb{E}\{q_{j,t}^2\} P_1 (B_j \otimes B_j) Q_2.$$

Suppose that for A_i, $i \in [r]$, *there exist* k_i, $l_i \in [n]$ *such that* $[A_i]_{k_i,l_i} \neq 0$ *and* $[A_j]_{k_i,l_i}$ *for all* $j \in [r] \setminus \{i\}$. *That is, the* (k_i, l_i)*-th entry of* A_i *is nonzero but the* (k_i, l_i)*-th entry of* A_j *is zero for all* $j \neq i$. *Then* $\mathbb{E}\{[\bar{A}_t]_{k_i,l_i}[\bar{A}_t]_{k_i,l_i}\} =$

$[A_i]_{k_i,l_i}^2 \mathbb{E}\{p_{i,t}^2\} = [A_i]_{k_i,l_i}^2 \sigma_i^2$. *From Proposition 1 we know that* $\sigma_i^2 = \mathbb{E}\{p_{i,t}^2\}$ *can be uniquely determined if second-moment dynamic (4), or* $\tilde{\Sigma}'_A$, *is given. A similar conclusion holds for* $\{q_{j,t}\}$. *However there are also situations where* σ_i^2 *cannot be uniquely determined. For instance, assume that* $r \geq 2$ *and for all* $i \in [r]$, $[A_i]_{11} \neq 0$ *but all other entries of* A_i *are zero. Then we only have a single equation* $\sum_{i=1}^r [A_i]_{11}^2 \sigma_i^2 = [\tilde{\Sigma}'_A]_{11}$ *for* $\{\sigma_i^2\}$.

As shown in Example 1, given (Σ_A, Σ_B) with $\Sigma_A \succeq 0$ and $\Sigma_B \succeq 0$, the set S_Σ^* is not empty but may have infinitely many elements, leading to an unidentifiable issue for entries $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\}$ and $\mathbb{E}\{[\bar{B}_t]_{ip}[\bar{B}_t]_{jq}\}$, $i \neq j$, $k \neq l$, $p \neq q$, $i,j,k,l \in [n]$, $p,q \in [m]$. The proposition below gives several conditions under which the covariance matrix of the multiplicative noise can be uniquely determined from $[\tilde{\Sigma}'_A \ \tilde{\Sigma}'_B]$ or not.

Proposition 2 *Given* (Σ_A, Σ_B) *with* $\Sigma_A \succeq 0$ *and* $\Sigma_B \succeq 0$, $\tilde{\Sigma}'_A = P_1 \Sigma'_A Q_1$ *and* $\tilde{\Sigma}'_B = P_2 \Sigma'_B Q_2$, *the following results hold.*
(i) If $n = m = 1$, *then* S_Σ^* *has a unique element. If* $m = 1$, *then* $S^*(\tilde{\Sigma}'_B)$ *has a unique element. If* $n \geq 2$ *and* $\Sigma_A \succ 0$ *(resp.* $m \geq 2$ *and* $\Sigma_B \succ 0$*), then* $S^*(\tilde{\Sigma}'_A)$ *(resp.* $S^*(\tilde{\Sigma}'_B)$*) has infinitely many elements, and as a result, under either condition,* S_Σ^* *has infinitely many elements.*
(ii) If $S^*(\tilde{\Sigma}'_A)$ *has infinitely many elements, then* $S^*(\tilde{\Sigma}'_A) \cap T_A$ *has a unique element, where*

$$T_A := \Big\{ \Sigma \in \mathbb{R}^{n^2 \times n^2} : \gamma_{ij,kl} \Sigma_{(k-1)n+i,(l-1)n+j} + \delta_{ij,kl} \Sigma_{(l-1)n+i,(k-1)n+j} = \tau_{ij,kl}, i<j, k<l, i,j,k,l \in [n] \Big\},$$

with fixed constants $\gamma_{ij,kl}, \delta_{ij,kl}, \tau_{ij,kl} \in \mathbb{R}$ *and* $\gamma_{ij,kl} \neq \delta_{ij,kl}$ *for all* $i<j$, $k<l$, $i,j,k,l \in [n]$. *The same result holds for* $S^*(\tilde{\Sigma}'_B)$ *by modifying the definition of* T_A *according to the dimension of* Σ_B.

PROOF. The first two conclusions of (i) are trivial. If $n \geq 2$, then $\tilde{\Sigma}'_A$ has entries of the form $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\} + \mathbb{E}\{[\bar{A}_t]_{il}[\bar{A}_t]_{jk}\}$. Since $\Sigma_A \succ 0$, its minimum eigenvalue is larger than zero. Note that from the definition of $S^*(\tilde{\Sigma}'_A)$ there exists α^* such that $\Sigma_A = \Sigma_A(\alpha^*)$. Because the eigenvalues of a matrix depend continuously on its entries (Theorem 2.4.9.2 of [37]), $\Sigma_A(\alpha^* + \varepsilon)$ is still a positive definite matrix for small enough $\varepsilon > 0$. This proves the last result in (i). From (i), we know that if $S^*(\tilde{\Sigma}'_A)$ has infinitely many elements, then $n \geq 2$. To show (ii), just note that if $\Sigma = \mathrm{vec}(\bar{A})\,\mathrm{vec}(\bar{A})^\intercal$ for some $A \in \mathbb{R}^{n \times n}$, then the $[(k-1)n+i,(l-1)n+j]$-th entry of Σ is $[\bar{A}]_{ik}[\bar{A}]_{jl}$ and the $[(l-1)n+i,(k-1)n+j]$-th entry is $[\bar{A}]_{il}[\bar{A}]_{jk}$, $i \neq j$, $k \neq l$. Hence if $\gamma_{ij,kl} \neq \delta_{ij,kl}$ then we have two linearly independent equations for $[\bar{A}]_{ik}[\bar{A}]_{jl}$ and $[\bar{A}]_{il}[\bar{A}]_{jk}$ (the other one from Proposition 1 is $[\bar{A}]_{ik}[\bar{A}]_{jl} + [\bar{A}]_{il}[\bar{A}]_{jk} = [\tilde{\Sigma}'_A]_{(i-1)(n-i/2)+j,(k-1)(n-k/2)+l}$). So these entries can be uniquely determined, and the conclusion follows.

Remark 2 *The first part of the above proposition shows that if* $\Sigma_A \succ 0$ *or* $\Sigma_B \succ 0$ *and* $n \geq m \geq 2$, *then it is impossible to uniquely determine* (Σ_A, Σ_B) *only based on second-moment dynamic (5). However the second part indicates that if we introduce more conditions for the covariance matrix, then all entries of* Σ_A *and* Σ_B *can be identified. The set* T_A *in fact introduces additional constraints for* $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\}$, $i \neq j$, $k \neq l$. *For example, if entries in* $\bar{A}_t$ *are mutually independent, then* Σ_A *is diagonal and all of its entries except* $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{ik}\}$, $i,k \in [n]$, *are zero. In this case, we have* $\mathbb{E}\{[\bar{A}_t]_{ik}[\bar{A}_t]_{jl}\} - \mathbb{E}\{[\bar{A}_t]_{il}[\bar{A}_t]_{jk}\} = 0$, $i \neq j$, $k \neq l$.

Now we are ready to propose our estimation algorithm. To follow the above discussion and simplify analysis, we consider a least-squares method based on the first- and second-moment dynamics (3) and (5). Since we do not have access to the exact moment dynamics, we need to average over multiple independent rollouts to estimate them. To guarantee persistently exciting inputs, one needs to design their first and second moments in advance, in either a deterministic or a stochastic way. For example, generate the two moments from standard Gaussian and Wishart distributions [38], respectively, or set them periodically. The initial states of different rollouts are assumed to be i.i.d. subject to a same distribution $\mathcal{X}_0$ with finite second moment (see Section 3.2.2). The overall algorithm is shown in Algorithm 1, where the superscript (k) represents the k-th rollout. It should be noted that Algorithm 1 is different from classic recursive identification algorithms. The recursive least-squares algorithm [14, 16], for example, uses only one trajectory of a system. On the contrary, Algorithm 1 is based on multiple trajectories with finite length.

Based on the estimates $\hat{\tilde{\Sigma}}'_A$ and $\hat{\tilde{\Sigma}}'_B$, we can obtain an estimate $\hat{S}_\Sigma^*$ of the equivalent class (7), via replacing $\tilde{\Sigma}'_A$ and $\tilde{\Sigma}'_B$ in the definition (7) by their estimates. If the linear matrix inequalities are infeasible, i.e., $\hat{S}_\Sigma^* = \emptyset$, then one can project the estimates onto the positive semidefinite cone. However this is unlikely to happen when n_r is large, because of the consistency of Algorithm 1, as shown in the next section.

Algorithm 1
Multiple-trajectory averaging least-squares (MALS)

1: **for** t from 0 to $\ell - 1$ **do**
2: Generate $\nu_t \in \mathbb{R}^m$ and $\bar{U}_t \in \mathbb{R}^{m\times m}$ with $\bar{U}_t \succeq 0$.
3: **end for**
4: **for** k from 1 to n_r **do**
5: Generate $x_0^{(k)}$ independently from the initial multivariate distribution $\mathcal{X}_0$.
6: **for** t from 0 to $\ell - 1$ **do**
7: Generate $u_t^{(k)}$ independently from a multivariate distribution with first moment ν_t and second central moment $\bar{U}_t$,
8: $x_{t+1}^{(k)} = (A + \bar{A}_t^{(k)})x_t^{(k)} + (B + \bar{B}_t^{(k)})u_t^{(k)}$.
9: **end for**
10: **end for**
11: **for** t from 0 to ℓ **do**
12: Compute

$$\begin{aligned}
\hat{\mu}_t &:= \frac{1}{n_r}\sum_{k=1}^{n_r} x_t^{(k)},\\
\hat{\tilde{X}}_t &:= \frac{1}{n_r} P_1 \operatorname{vec}\left(\sum_{k=1}^{n_r} x_t^{(k)}(x_t^{(k)})^\intercal\right),\\
\hat{W}_t &:= \frac{1}{n_r}\operatorname{vec}\left(\sum_{k=1}^{n_r} x_t^{(k)}\nu_t^\intercal\right) = \operatorname{vec}(\hat{\mu}_t\nu_t^\intercal),\\
\hat{W}_t' &:= \frac{1}{n_r}\operatorname{vec}\left(\sum_{k=1}^{n_r} \nu_t x_t^{(k)\intercal}\right) = \operatorname{vec}(\nu_t\hat{\mu}_t^\intercal),\\
\tilde{U}_t &:= P_2 \operatorname{vec}(\bar{U}_t + \nu_t\nu_t^\intercal).
\end{aligned}$$

13: **end for**
14: $(\hat{A}, \hat{B}) = \operatorname*{argmin}_{(A,B)} \left\{\sum_{t=0}^{\ell-1} \|\hat{\mu}_{t+1} - (A\hat{\mu}_t + B\nu_t)\|_2^2\right\}$,
15: Compute $\hat{\tilde{A}} = P_1(\hat{A}\otimes\hat{A})Q_1$, $\hat{\tilde{B}} = P_1(\hat{B}\otimes\hat{B})Q_2$, $\hat{K}_{BA} = P_1(\hat{B}\otimes\hat{A})$, and $\hat{K}_{AB} = P_1(\hat{A}\otimes\hat{B})$, where P_1, P_2, Q_1, and Q_2 are given before (5),
16: $(\hat{\tilde{\Sigma}}_A', \hat{\tilde{\Sigma}}_B') = \operatorname*{argmin}_{(\tilde{\Sigma}_A',\tilde{\Sigma}_B')} \left\{\sum_{t=0}^{\ell-1} \|\hat{\tilde{X}}_{t+1} - [\tilde{A}\hat{\tilde{X}}_t + K_{BA}\hat{W}_t + K_{AB}\hat{W}_t' + \tilde{B}\tilde{U}_t + \tilde{\Sigma}_A'\hat{\tilde{X}}_t + \tilde{\Sigma}_B'\tilde{U}_t]\|_2^2\right\}$.

3.2 Consistency of Algorithm 1

In this section we analyze the consistency of Algorithm 1 by investigating the moment dynamics (3) and (5).

3.2.1 Moment Dynamics

Note again if we know μ_t and $\tilde{X}_t$, then it is possible to recover the parameters via least-squares as in lines 14-16 in Algorithm 1. Let

$$\mathbf{Y} := [\mu_\ell \ \cdots \ \mu_1],\ \mathbf{Z} := \begin{bmatrix}\mu_{\ell-1} & \cdots & \mu_0\\ \nu_{\ell-1} & \cdots & \nu_0\end{bmatrix},\ \mathbf{C} := [C_\ell \ \cdots \ C_1],\ \mathbf{D} := \begin{bmatrix}\tilde{X}_{\ell-1} & \cdots & \tilde{X}_0\\ \tilde{U}_{\ell-1} & \cdots & \tilde{U}_0\end{bmatrix}, \tag{8}$$

where $C_t = \tilde{X}_t - \big(\tilde{A}\tilde{X}_{t-1} + K_{BA}W_{t-1} + K_{AB}W_{t-1}' + \tilde{B}\tilde{U}_{t-1}\big)$, $1 \le t \le \ell$. Then closed-form solutions of the least-squares problems are

$$\begin{bmatrix}\hat{A} & \hat{B}\end{bmatrix} = \mathbf{Y}\mathbf{Z}^\intercal(\mathbf{Z}\mathbf{Z}^\intercal)^\dagger,\ \begin{bmatrix}\hat{\tilde{\Sigma}}_A' & \hat{\tilde{\Sigma}}_B'\end{bmatrix} = \mathbf{C}\mathbf{D}^\intercal(\mathbf{D}\mathbf{D}^\intercal)^\dagger,$$

where $\mathbf{C}$, $\mathbf{D}$, $\mathbf{Y}$, and $\mathbf{Z}$ are defined in (8), and the sign † represents the pseudoinverse. When the inverse matrices exist, the solutions are identical to true values, that is, $[\hat{A}\ \hat{B}] = [A\ B]$ and $[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B] = [\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]$. Hence, the first question towards the consistency of Algorithm 1 is whether the matrices $\mathbf{Z}\mathbf{Z}^\intercal$ and $\mathbf{D}\mathbf{D}^\intercal$ are invertible. As to be shown, this invertibility can be obtained by designing a proper input sequence, if systems (A, B) and $(\tilde{A} + \tilde{\Sigma}'_A, \tilde{B} + \tilde{\Sigma}'_B)$ are controllable, and the rollout length ℓ is large enough.

Proposition 3 *Suppose that $\ell \geq n + m$ and (A, B) is controllable. For fixed $\mu_0 \in \mathbb{R}^n$, the matrix $\mathbf{Z}$ has full row rank, and consequently $\mathbf{Z}\mathbf{Z}^\intercal$ is invertible, for almost all $[\nu_0^\intercal\ \cdots\ \nu_{\ell-1}^\intercal]^\intercal \in \mathbb{R}^{m\ell}$.*

PROOF. See Appendix A.

Remark 3 *The above proposition shows that for large enough time step of each rollout, the full row rank of $\mathbf{Z}$ can be guaranteed for almost all $[\nu_0^\intercal\ \cdots\ \nu_{\ell-1}^\intercal]^\intercal \in \mathbb{R}^{m\ell}$. In the proof, the controllability of (A, B) plays a key role, similar to classic results on identification of linear systems [16]. The condition $\ell \geq n + m$ is necessary for the invertibility of $\mathbf{Z}\mathbf{Z}^\intercal$. According to the proposition, $\mathbf{Z}\mathbf{Z}^\intercal$ is invertible with probability one if one randomly generates the first moments of inputs i.i.d. from a distribution absolutely continuous with respect to Lebesgue measure, e.g., Gaussian distribution or uniform distribution. This proposition can be seen as a generalization of the single-input case in [39].*

Proposition 4 *Suppose that $\ell \geq [n(n+1) + m(m+1)]/2$ and $(\tilde{A} + \tilde{\Sigma}'_A, \tilde{B} + \tilde{\Sigma}'_B)$ is controllable. For fixed $\mu_0 \in \mathbb{R}^n$ and $\tilde{X}_0 \in \mathbb{R}^{n(n+1)/2}$, the matrix $\mathbf{D}$ has full row rank, and consequently $\mathbf{D}\mathbf{D}^\intercal$ is invertible, for almost all $[\nu_0^\intercal\ \cdots\ \nu_{\ell-1}^\intercal\ \operatorname{svec}(\bar{U}_0)^\intercal\ \cdots\ \operatorname{svec}(\bar{U}_{\ell-1}))^\intercal]^\intercal \in \mathbb{R}^{\ell m(m+3)/2}$, where $\bar{U}_t$ is defined in line 2 of Algorithm 1.*

PROOF. See Appendix B.

Remark 4 *The controllability condition in Proposition 4 reflects the nature of the multiplicative noise, i.e., coupling between $\bar{A}_t$ and x_t, and that between $\bar{B}_t$ and u_t. It also indicates that a controllability condition on (5) is necessary to ensure the successful identification of the covariance matrix. The condition $\ell \geq [n(n+1) + m(m+1)]/2$ is necessary for the invertibility of $\mathbf{D}\mathbf{D}^\intercal$. As in Algorithm 1, $\tilde{U}_t = \operatorname{svec}(\bar{U}_t + \nu_t\nu_t^\intercal)$, so random generation of ν_t and $\bar{U}_t$ can ensure $\mathbf{D}\mathbf{D}^\intercal$ is invertible with probability one.*

Corollary 1 *Suppose that $\ell \geq [n(n+1) + m(m+1)]/2$, and both (A, B) and $(\tilde{A} + \tilde{\Sigma}'_A, \tilde{B} + \tilde{\Sigma}'_B)$ are controllable. For fixed $\mu_0 \in \mathbb{R}^n$ and $\tilde{X}_0 \in \mathbb{R}^{n(n+1)/2}$, the matrices $\mathbf{Z}\mathbf{Z}^\intercal$ and $\mathbf{D}\mathbf{D}^\intercal$ are invertible, for almost all $[\nu_0^\intercal\ \cdots\ \nu_{\ell-1}^\intercal\ \operatorname{svec}(\bar{U}_0)^\intercal\ \cdots\ \operatorname{svec}(\bar{U}_{\ell-1}))^\intercal]^\intercal \in \mathbb{R}^{\ell m(m+3)/2}$, where $\bar{U}_t$ is defined in line 2 of Algorithm 1.*

Remark 5 *From the proof of Propositions 3 and 4, we know that the existence of $(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}$ and $(\mathbf{D}\mathbf{D}^\intercal)^{-1}$ can indeed be guaranteed with probability one, as long as ν_t and $\bar{U}_t$, the mean and covariance matrix of the input at time t, are independently generated from distributions that is absolutely continuous with respect to Lebesgue measure. For example, the entries of ν_t are generated i.i.d. from a non-degenerate Gaussian distribution and then $\bar{U}_t$ is generated i.i.d. from a non-degenerate Wishart distribution, $0 \leq t \leq \ell - 1$.*

3.2.2 Consistency

After the discussion in the previous section, we now assume that the expectations and covariance matrices of inputs have been generated, and both $\mathbf{Z}\mathbf{Z}^\intercal$ and $\mathbf{D}\mathbf{D}^\intercal$ have been designed to be invertible. The closed-form estimates generated by Algorithm 1 are

$$\begin{bmatrix}\hat{A}\ \hat{B}\end{bmatrix} = \hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger, \tag{9}$$

$$\begin{bmatrix}\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B\end{bmatrix} = \hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^\dagger, \tag{10}$$

where

$$\hat{\mathbf{Y}} := \begin{bmatrix}\hat{\mu}_\ell & \cdots & \hat{\mu}_1\end{bmatrix}, \quad \hat{\mathbf{Z}} := \begin{bmatrix}\hat{\mu}_{\ell-1} & \cdots & \hat{\mu}_0 \\ \nu_{\ell-1} & \cdots & \nu_0\end{bmatrix}, \tag{11}$$

$$\hat{\mathbf{C}} := \begin{bmatrix}\hat{C}_\ell & \cdots & \hat{C}_1\end{bmatrix}, \quad \hat{\mathbf{D}} := \begin{bmatrix}\hat{\tilde{X}}_{\ell-1} & \cdots & \hat{\tilde{X}}_0 \\ \tilde{U}_{\ell-1} & \cdots & \tilde{U}_0\end{bmatrix}, \tag{12}$$

and $\hat{C}_t = \hat{\tilde{X}}_t - \big(\hat{\tilde{A}}\hat{\tilde{X}}_{t-1} + \hat{K}_{BA}\hat{W}_{t-1} + \hat{K}_{AB}\hat{W}'_{t-1} + \hat{\tilde{B}}\tilde{U}_{t-1}\big)$, $1 \le t \le \ell$. Here $\hat{\tilde{A}}$, $\hat{\tilde{B}}$, $\hat{K}_{AB}$, and $\hat{K}_{BA}$ are estimates of $\tilde{A}$, $\tilde{B}$, K_{AB}, and K_{BA}, obtained from $\hat{A}$ and $\hat{B}$ given by Algorithm 1. The estimates above depend on the number of rollouts n_r, but we omit it for convenience. Before stating the convergence result, we present the following assumptions.

Assumption 1 *For all rollouts indexed by $1 \le k \le n_r$, the below conditions hold.*
(i) The rollout length is $\ell \ge [n(n+1) + m(m+1)]/2$.
(ii) The initial states $x_0^{(k)}, 1 \le k \le n_r$, are i.i.d. subject to the same distribution $\mathcal{X}_0$ with finite second moment, and are independent of the multiplicative noise and inputs.
(iii) $\{\bar{A}_t^{(k)}, 0 \le t \le \ell, 1 \le k \le n_r\}$ and $\{\bar{B}_t^{(k)}, 0 \le t \le \ell, 1 \le k \le n_r\}$, are i.i.d. sequences respectively and are mutually independent, both with zero mean and finite second moments; i.e., $\mathbb{E}\{\bar{A}_t^{(k)}\}$ and $\mathbb{E}\{\bar{B}_t^{(k)}\}$ are zero matrices, and $\|\Sigma_A\|_2, \|\Sigma_B\|_2 < \infty$.
(iv) The parameters of inputs are given by lines 1-3 of Algorithm 1, and the inputs are generated, according to line 7 of Algorithm 1, independently of the multiplicative noise.
(v) Both $\mathbf{Z}\mathbf{Z}^\intercal$ and $\mathbf{D}\mathbf{D}^\intercal$ are invertible.

Under Assumption 1 the rollouts $\{x_0^{(k)}, \ldots, x_l^{(k)}\}$, $1 \le k \le n_r$, are i.i.d., so consistency can be established from strong law of large numbers.

Theorem 1 *(Consistency) Suppose that Assumption 1 holds, then the estimators (9)-(10) are asymptotically consistent, i.e.,*

$$\begin{bmatrix}\hat{A} \ \hat{B}\end{bmatrix} \to \begin{bmatrix}A \ B\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}\hat{\tilde{\Sigma}}'_A \ \hat{\tilde{\Sigma}}'_B\end{bmatrix} \to \begin{bmatrix}\tilde{\Sigma}'_A \ \tilde{\Sigma}'_B\end{bmatrix},$$

with probability one as the number of rollouts $n_r \to \infty$.

PROOF. See Appendix C.

Remark 6 *This theorem indicates that a relatively small rollout length suffices, while consistency is guaranteed by an increasing number of rollouts. In [29], the estimation of the first and second moments of multiplicative noise is decoupled, while here the estimate of $[\tilde{\Sigma}'_A \ \tilde{\Sigma}'_B]$ relies on $[\hat{A} \ \hat{B}]$. This is because we estimate the covariance matrix of the noise, which from definition depends on its first moment. Note that we assume that ℓ is fixed and do not consider the case where $\ell \to \infty$, since an averaging step is used in Algorithm 1, to simplify the analysis. Study of the case with increasing rollout length is left to future work.*

3.3 Finite-Sample Analysis

In this subsection we present a finite-sample result of Algorithm 1, demonstrating its non-asymptotic behavior. The existence of multiplicative noise complicates the analysis, so here we introduce additional assumptions for simplicity to ensure that the states are bounded a.s.

Assumption 2 *For all rollouts indexed by $1 \le k \le n_r$, the following conditions hold.*
(i) The initial state is bounded a.s. for all rollouts $1 \le k \le n_r$ as

$$\|x_0^{(k)}\| \le c_X < \infty.$$

(ii) The inputs are bounded a.s. for all $0 \le t \le \ell - 1$ and $1 \le k \le n_r$ as

$$\|u_t^{(k)}\| \le c_U < \infty.$$

(iii) The multiplicative noise, $\bar{A}_t^{(k)}$ and $\bar{B}_t^{(k)}$, is bounded a.s. for all $0 \le t \le \ell - 1$ and $1 \le k \le n_r$ as

$$\|\bar{A}_t^{(k)}\|_2 \le c_{\bar{A}} < \infty, \ \|\bar{B}_t^{(k)}\|_2 \le c_{\bar{B}} < \infty.$$

Remark 7 *The assumption of bounded multiplicative noise is reasonable for physical systems, which cannot have infinite variations. For example, in interconnected systems, the noise represents randomly varying topologies of subsystems, and is naturally bounded.*

Introduce the state- and input-deviation quantities

$$\begin{aligned} e_t^{(k)} &:= x_t^{(k)} - \mathbb{E}[x_t^{(k)}] = x_t^{(k)} - \mu_t, \\ d_t^{(k)} &:= u_t^{(k)} - \mathbb{E}[u_t^{(k)}] = u_t^{(k)} - \nu_t. \end{aligned}$$

The next proposition is a natural consequence of Assumption 2.

Proposition 5 *Under Assumption 2, the following results hold.*
(i) The initial state-deviation is bounded a.s. for all rollouts $1 \le k \le n_r$ as

$$\|e_0^{(k)}\| \le c_\mu < \infty.$$

(ii) The outer product initial state deviation is bounded a.s. for all rollouts $1 \le k \le n_r$ as

$$\left\|\mathrm{vec}\left(x_0^{(k)}(x_0^{(k)})^\intercal - \mathbb{E}\left[x_0^{(k)}(x_0^{(k)})^\intercal\right]\right)\right\| \le c_{\Delta X}.$$

(iii) The input-deviations are bounded a.s. for all $0 \le t \le \ell - 1$ and $1 \le k \le n_r$ as

$$\|d_t^{(k)}\| \le c_\nu < \infty.$$

(iv) The Kronecker products of $\bar{A}_t^{(k)}$ and $\bar{B}_t^{(k)}$ are bounded a.s. for all $0 \le t \le \ell - 1$ and $1 \le k \le n_r$ as

$$\begin{aligned} \|\bar{A}_t^{(k)} \otimes \bar{A}_t^{(k)} - \Sigma'_A\|_2 &\le c_{\Sigma'_A}, \\ \|\bar{B}_t^{(k)} \otimes \bar{B}_t^{(k)} - \Sigma'_B\|_2 &\le c_{\Sigma'_B}. \end{aligned}$$

Remark 8 *This proposition captures the deviations of random components of System (1) from their expectations. Using the bounds in Assumption 2 one could upper-bound these deviations, for instance,*

$$\left\|\mathrm{vec}\left(x_0^{(k)}(x_0^{(k)})^\intercal - \mathbb{E}\left[x_0^{(k)}(x_0^{(k)})^\intercal\right]\right)\right\| \le \left\|x_0^{(k)}(x_0^{(k)})^\intercal\right\|_F + \left\|\mathbb{E}\{x_0^{(k)}(x_0^{(k)})^\intercal\}\right\|_F \le 2c_X^2,$$

and

$$\begin{aligned} \|\bar{A}_t^{(k)} \otimes \bar{A}_t^{(k)} - \Sigma'_A\|_2 \le \|\bar{A}_t^{(k)} \otimes \bar{A}_t^{(k)}\|_2 + \|\Sigma'_A\|_2 = \|\bar{A}_t^{(k)}\|_2^2 + \|\Sigma'_A\|_2 \le c_{\bar{A}}^2 + \|\Sigma'_A\|_2, \\ \|\bar{B}_t^{(k)} \otimes \bar{B}_t^{(k)} - \Sigma'_B\|_2 \le \|\bar{B}_t^{(k)} \otimes \bar{B}_t^{(k)}\|_2 + \|\Sigma'_B\|_2 = \|\bar{B}_t^{(k)}\|_2^2 + \|\Sigma'_B\|_2 \le c_{\bar{B}}^2 + \|\Sigma'_B\|_2. \end{aligned}$$

However, these bounds may not depend on those in Assumption 2. For example, when $x_0^{(k)}$ is a nonzero constant, $c_\mu = 0$ but c_X is positive.

The boundedness of the states and state-deviations follows from Assumptions 1 and 2 according to the below statement.

Lemma 1 *Suppose that Assumptions 1 and 2 hold, then for all $1 \le k \le n_r$ and $0 \le t \le \ell$ we have that*

$$\left\|x_t^{(k)}\right\| \le c_M, \quad \left\|e_t^{(k)}\right\| \le c_N, \quad \left\|x_t^{(k)}(x_t^{(k)})^\intercal\right\|_2 \le c_M^2, \quad \left\|e_t^{(k)}(e_t^{(k)})^\intercal\right\|_2 \le c_N^2,$$

and

$$\left\|\mathrm{vec}\left(x_t^{(k)}(x_t^{(k)})^\intercal - \mathbb{E}\left\{x_t^{(k)}(x_t^{(k)})^\intercal\right\}\right)\right\| \leq c_F, \quad \left\|\mathrm{vec}\left(x_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\left\{x_t^{(k)}(u_t^{(k)})^\intercal\right\}\right)\right\| \leq c_W,$$

where

$$\begin{aligned}
c_M &:= \max_{0\leq t\leq \ell}\left\{c_A^t c_X + \sum_{i=0}^{t-1} c_A^i c_B c_U\right\},\\
c_N &:= \max_{0\leq t\leq \ell}\left\{\|A\|_2^t c_\mu + \sum_{i=0}^{t-1}\|A\|_2^i(\|B\|_2 c_\nu + c_{\bar{A}} c_M + c_{\bar{B}} c_U)\right\},\\
c_F &:= \max_{0\leq t\leq \ell}\left\{(\|A\|_2^2 + \|\Sigma'_A\|_2)^t c_{\Delta X} + \sum_{i=0}^{t-1}(\|A\|_2^2 + \|\Sigma'_A\|_2)^i(c_{FX} + c_{FU} + c_{FXU})\right\},\\
c_W &:= c_N c_U + c_M c_\nu,
\end{aligned}$$

and

$$\begin{aligned}
c_A &:= \|A\|_2 + c_{\bar{A}},\\
c_B &:= \|B\|_2 + c_{\bar{B}},\\
c_{FX} &:= (2\|A\|_2 c_{\bar{A}} + c_{\Sigma'_A})c_M^2,\\
c_{FU} &:= 3(\|B\|_2^2 + \|\Sigma'_B\|_2)c_U c_\nu + (2\|B\|_2 c_{\bar{B}} + c_{\Sigma'_B})c_U^2,\\
c_{FXU} &:= 2\|A\|_2\|B\|_2 c_W + (2\|A\|_2 c_{\bar{B}} + 2\|B\|_2 c_{\bar{A}} + 2c_{\bar{A}} c_{\bar{B}})c_M c_U.
\end{aligned}$$

PROOF. See Appendix D.

Remark 9 *The quantity c_M can be interpreted as a bound on the radius from the origin to the outer boundary of the set of reachable states from any valid x_0 over ℓ time steps. If the system is not robustly stable in the sense that $c_A > 1$, then the limit as $\ell \to \infty$ of c_M could be infinite. However, since we consider only finite-length rollouts, c_M is finite regardless of the stability properties of the system.*

Analogous interpretations follow for the quantity c_N and the reachable state-deviations. Notice that the constants c_N grows with increasing maximum initial state and input deviations c_μ and c_ν, and maximum noise magnitudes $c_{\bar{A}}$ and $c_{\bar{B}}$. Conversely, c_N vanishes as those quantities become smaller, i.e. in the case that the initial state x_0 is a fixed deterministic value, the inputs u_t follow a deterministic sequence, and there is no multiplicative noise. Likewise, c_F vanishes in such a scenario, so that $c_{\Delta X} = c_{FX} = c_{FU} = c_{FXU} = 0$.

We have the following finite-sample result for the estimates of $[A\ B]$ and $[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]$, whose proofs are given in Appendices F and G respectively.

Theorem 2 *Suppose that Assumptions 1-2 hold. Fix a failure probability $\delta \in (0,1)$. It holds with probability at least $1-\delta$ that*

$$\left\|[\hat{A}\ \hat{B}] - [A\ B]\right\|_2 = \mathcal{O}\left(\sqrt{\frac{\ell\log(\ell/\delta)}{n_r}}\right).$$

Theorem 3 *Under the same condition of Theorem 2, with probability at least $1-\delta$, it holds that*

$$\left\|\left[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B\right] - \left[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B\right]\right\|_2 = \mathcal{O}\left(\sqrt{\frac{\ell\log(\ell/\delta)}{n_r}}\right).$$

Remark 10 *In Theorems 2 and 3 qualitative high-probability upper bounds are given for the estimates of* $[A\ B]$ *and* $[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]$. *It can be observed that these bounds converge to zero as the number of rollouts,* n_r, *grows to infinity, indicating the consistency of the estimator. It should be noted that the bounds are deterministic, though they depend on the failure probability* δ. *The theorems also indicate that given a positive constant, the probability of the estimation error exceeding this constant decays exponentially fast as* n_r *increases, which is illustrated in Section 4.1.*

The $\mathcal{O}(\cdot)$ *notation hides the coefficients of the error bounds, and the polynomial and exponential factors of* n *and* m *in the logarithm term. Their explicit forms are given in Appendices F and G. The coefficient of the estimation error of* $[A\ B]$ *increases with* $\|\mathbf{Y}\|_2$, $\|\mathbf{Z}\|_2$, *and the bound of the system, but decreases with the minimum eigenvalue of* $\mathbf{ZZ}^\intercal$. *Similarly, the coefficient of the estimation error of* $[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]$ *decreases with the minimum eigenvalue of* $\mathbf{DD}^\intercal$, *but increases with* $\|\mathbf{C}\|_2$, $\|\mathbf{D}\|_2$, *and the bound of the system. It also increases with* $\|A\|_2$, $\|B\|_2$, *and quantities related to the second-moment dynamic of system states, due to the dependence of* $[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B]$ *on* $[\hat{A}\ \hat{B}]$. *From definition,* $\mathbf{Y}$, $\mathbf{Z}$, $\mathbf{C}$, *and* $\mathbf{D}$ *depend on the parameters of the system and inputs, so proper input design could reduce the estimation error. But since the nominal matrix is unknown, optimal input design strategies and data-dependent bounds still need to be studied.*

Note that the current bounds imply that longer rollout length ℓ *leads to worse performance, which seems to be contrary to the intuition that longer trajectory provides more data. This could result from the averaging step which eliminates some excitation. Future work will consider how to use the data more efficiently.*

4 Numerical Simulations

In this section we empirically validate the theoretical results for Algorithm 1, and compare its performance with the recursive least-squares algorithm [14, 16].

4.1 Consistency and Finite-Sample Result

We continue to consider Example 1, with parameters as follows,

$$A = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix},\ \Sigma_A = \frac{1}{40}\begin{bmatrix} 8 & -2 & 0 & 0 \\ -2 & 16 & 2 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 0 & 8 \end{bmatrix},\ \Sigma_B = \frac{1}{40}\begin{bmatrix} 5 & -2 \\ -2 & 20 \end{bmatrix}.$$

According to the reshaping operator G defined in Section 1.4 and Example 1, we have

$$\tilde{\Sigma}'_A = \frac{1}{40}\begin{bmatrix} 8 & 0 & 2 \\ -2 & 2 & 0 \\ 16 & 0 & 8 \end{bmatrix},\ \tilde{\Sigma}'_B = \frac{1}{40}\begin{bmatrix} 5 & -2 & 20 \end{bmatrix}^\intercal. \tag{13}$$

A simulated experiment is conducted with rollout data of length $\ell = [n(n+1) + m(m+1)]/2 = 4$. For $0 \le t \le 3$, ν_t is generated independently from uniform distribution $\mathcal{U}([0,1])$ and then fixed. Three types of inputs are considered: Gaussian, uniform, and deterministic inputs. An identical sequence of input covariances, independently generated from one-dimensional Wishart distribution $W_p(0.1, 1)$ and then fixed, are used in the former two cases. For the case of deterministic inputs, the covariances are set to be zero, i.e., $\bar{U}_t = 0$. This is able to make $\mathbf{DD}^\intercal$ invertible because the second moment of the input at time t satisfies that $U_t = \bar{U}_t + \nu_t\nu_t^\intercal$, and the generation of the latter provides randomness. For each case, Algorithm 1 is run for 50 times. The mean of estimation error in each case is shown in Fig. 1. It can be seen that Algorithm 1 converges and performs similarly under all three types of control inputs, with convergence rate $\mathcal{O}(1/\sqrt{n_r})$. The algorithm fluctuates when the number of rollouts is small, which may result from error arising in averaging trajectories.

Fig. 2 provides the relative frequency of the normalized estimates, $\|[\hat{A}\ \hat{B}] - [A\ B]\|_2 / \|[A\ B]\|_2$ and $\|[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B] - [\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]\|_2 / \|[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B]\|_2$, exceeding a given constant, under the uniform input case. This indicates an exponential decay and validates the finite-sample result. The relative frequence of n_r rollouts is denoted by p_{n_r}.

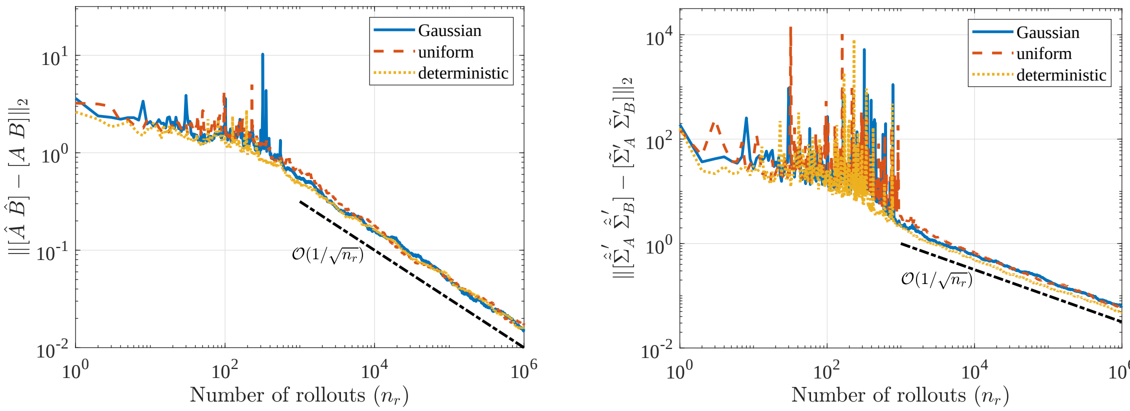

Fig. 1. Consistency of Algorithm 1.

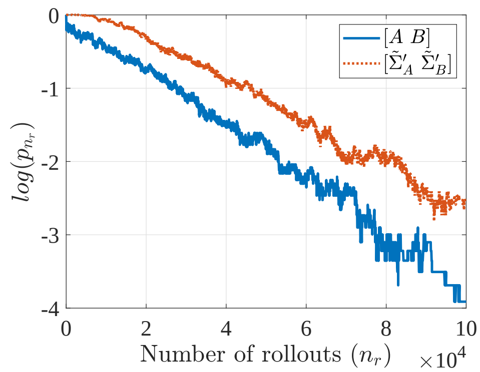

Fig. 2. Finite-sample result of Algorithm 1.

From Remark 1 and (7), (13) defines an equivalent class of covariance matrices that generates the same second-moment dynamic of system states. In the current example, Σ_B is unique, but the following covariance matrix is equivalent to Σ_A in the previously discussed sense,

$$\Sigma_A(\alpha) = \frac{1}{40}\begin{bmatrix} 8 & -2 & 0 & 1+\alpha \\ -2 & 16 & 1-\alpha & 0 \\ 0 & 1-\alpha & 2 & 0 \\ 1+\alpha & 0 & 0 & 8 \end{bmatrix},$$

where $\alpha \in \mathbb{R}$ and is such that $\Sigma_A(\alpha) \succeq 0$. In Fig. 3, we illustrate the dynamic (4) given by (Σ_A, Σ_B), $(\Sigma_A(1), \Sigma_B)$, and the estimated parameters of Algorithm 1 respectively, starting with the same initial condition $\mu_0 = \mathbf{0}_2$ and $X_0 = \mathbf{0}_4$. The parameters of inputs, ν_t and $\bar{U}_t$, are the same as the uniform input case above. Note that $\Sigma_A(-1) = \Sigma_A$, and $\Sigma_A(1) \succ 0$. It can be observed that the dynamics given by (Σ_A, Σ_B) and $(\Sigma_A(1), \Sigma_B)$ are identical, and the dynamic given by the estimates from Algorithm 1 is close to the former.

It is assumed that there is no additive noise in System (1), but Algorithm 1 can also be applied to identify linear systems with both multiplicative and additive noise. If additive noise w_t, independent of the inputs and the multiplicative noise, exists, then we can write the system as

$$\begin{aligned} x_{t+1} &= (A + \bar{A}_t)x_t + (B + \bar{B}_t)u_t + w_t \\ &= (A + \bar{A}_t)x_t + \begin{bmatrix} B + \bar{B}_t & w_t \end{bmatrix}\begin{bmatrix} u_t \\ 1 \end{bmatrix}. \end{aligned} \tag{14}$$

This means that we consider w_t as a part of multiplicative noise corresponding to a constant input with value one.

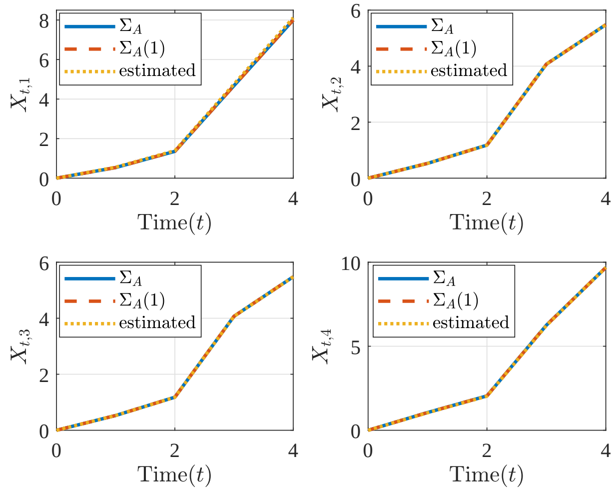

Fig. 3. The second-moment dynamic of system states given by several noise covariance matrices.

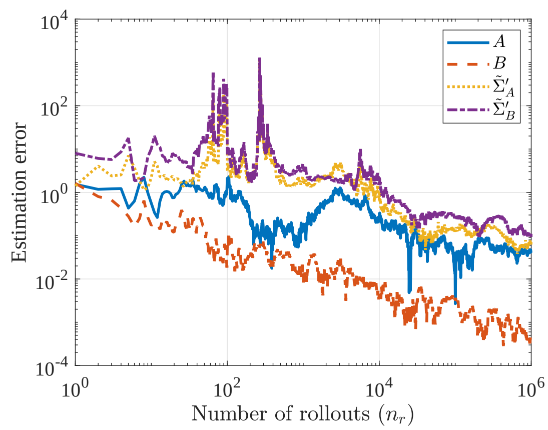

Fig. 4. Consistency of Algorithm 1 under both multiplicative and additive noise.

Consider the above 2-dimensional system with $w_t \sim \mathcal{N}(\mathbf{0}_2, \sigma^2 I_2)$, i.e., Gaussian noise with zero mean and $\sigma = 0.2$, and Gaussian inputs designed as above. Note that in this case we need $\ell = 6$ because the dimension of inputs increases by one in (14), compared with the original system. Fig. 4 shows the consistency of Algorithm 1 under the presence of additive noise.

4.2 Performance Comparison

The recursive form of ordinary least-squares (OLS), i.e., recursive least-squares (RLS), is widely used in identification of dynamic systems [14, 16]. It is possible to apply RLS to identify System (1) if certain conditions hold. Note that from System (1), we have that

$$\begin{aligned} x_{t+1} &= Ax_t + Bu_t + (\bar{A}_t x_t + \bar{B}_t u_t) \\ &= Ax_t + Bu_t + w_t^{(1)}, \end{aligned}$$

where $w_t^{(1)} := \bar{A}_t x_t + \bar{B}_t u_t$ is considered to be noise. Under Assumption 1, $\{w_t^{(1)}, \mathcal{F}_t\}$ is a martingale difference sequence, i.e., $\mathbb{E}\{w_t^{(1)} | \mathcal{F}_{t-1}\} = 0$, where $\mathcal{F}_t := \sigma(\bar{A}_k, \bar{B}_k, u_k, 0 \le k \le t)$. A mild condition for $w_t^{(1)}$ to ensure convergence of RLS in literature [14, 16] is $\sup_t \mathbb{E}\{\|w_t^{(1)}\|^\beta | \mathcal{F}_{t-1}\} < \infty$ a.s. for some $\beta > 2$. However in our case $w_t^{(1)}$ is state-dependent, so certain stability assumption is needed to ensure this boundedness condition. This means that RLS could fail if the nominal part of System (1) is marginally stable ($\rho(A) \le 1$) or unstable ($\rho(A) > 1$). In contrast, Algorithm 1 can handle this situation with the help of multiple trajectory data. Similarly, the noise covariance matrix of System (1) may be estimated using the following dynamic

$$
\begin{aligned}
&P_1 \operatorname{vec}(x_{t+1}x_{t+1}^\intercal) \\
&= P_1\big((A+\bar{A}_t)\otimes(A+\bar{A}_t)\big)Q_1P_1\operatorname{vec}(x_tx_t^\intercal) + P_1\big((B+\bar{B}_t)\otimes(B+\bar{B}_t)\big)Q_2P_2\operatorname{vec}(u_tu_t^\intercal) \\
&\quad + P_1\big((B+\bar{B}_t)\otimes(A+\bar{A}_t)\big)\operatorname{vec}(x_tu_t^\intercal) + P_1\big((A+\bar{A}_t)\otimes(B+\bar{B}_t)\big)\operatorname{vec}(u_tx_t^\intercal) \\
&= P_1\mathbb{E}\big\{(A+\bar{A}_t)\otimes(A+\bar{A}_t)\big\}Q_1P_1\operatorname{vec}(x_tx_t^\intercal) + P_1\mathbb{E}\big\{(B+\bar{B}_t)\otimes(B+\bar{B}_t)\big\}Q_2P_2\operatorname{vec}(u_tu_t^\intercal) \\
&\quad + P_1(B\otimes A)\operatorname{vec}(x_tu_t^\intercal) + P_1(A\otimes B)\operatorname{vec}(u_tx_t^\intercal) + w_t^{(2)},
\end{aligned}
$$

where

$$
\begin{aligned}
w_t^{(2)} := {}& P_1\big((A+\bar{A}_t)\otimes(A+\bar{A}_t) - \mathbb{E}\{(A+\bar{A}_t)\otimes(A+\bar{A}_t)\}\big)Q_1P_1\operatorname{vec}(x_tx_t^\intercal) \\
&+ P_1\big((B+\bar{B}_t)\otimes(B+\bar{B}_t) - \mathbb{E}\{(B+\bar{B}_t)\otimes(B+\bar{B}_t)\}\big)Q_2P_2\operatorname{vec}(u_tu_t^\intercal) \\
&+ P_1(B\otimes\bar{A}_t + \bar{B}_t\otimes A + \bar{B}_t\otimes\bar{A}_t)\operatorname{vec}(x_tu_t^\intercal) \\
&+ P_1(A\otimes\bar{B}_t + \bar{A}_t\otimes B + \bar{A}_t\otimes\bar{B}_t)\operatorname{vec}(u_tx_t^\intercal).
\end{aligned}
$$

One can obtain that under Assumption 1, though state-dependent, $\{w_t^{(2)}, \mathcal{F}_t\}$ is also a martingale difference sequence. To estimate the covariance matrix of the multiplicative noise, [29] applies OLS, equivalent to RLS, to the above dynamic. It should be noted that when utilizing OLS/RLS, one estimates the second moments of $A+\bar{A}_t$ and $B+\bar{B}_t$, rather than their covariance matrices, Σ_A and Σ_B in our context. The estimation of noise covariance is still coupled with the estimation of the nominal system, since $\Sigma'_A = \mathbb{E}\{(A+\bar{A}_t)\otimes(A+\bar{A}_t)\} - A\otimes A$ and $\Sigma'_B = \mathbb{E}\{(B+\bar{B}_t)\otimes(B+\bar{B}_t)\} - B\otimes B$.

To compare the performance of RLS and Algorithm 1, we consider four systems. In the first case, let the nominal system matrices be

$$
A = \begin{bmatrix} 0.6 & 0.2 \\ 0 & 0.6 \end{bmatrix}, \ B = \begin{bmatrix} 0.8 \\ 1 \end{bmatrix},
$$

and $\Sigma_A = \Sigma_B$ be all-zero matrices. That is, a linear system without noise and $\rho(A) = 0.6$. We use this case to show the consistency of RLS. In the other three cases, the matrix A is set to be $\begin{bmatrix} 0.6 & 0.2 \\ 0 & 0.6 \end{bmatrix}$, $\begin{bmatrix} 0.8 & 0.2 \\ 0 & 0.8 \end{bmatrix}$, and $\begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}$, respectively. B is the same as the first case, while Σ_A and Σ_B in Section 4.1 are adopted to be the covariance matrices. The implementation of Algorithm 1 is the same as in Section 4.1. That is, ν_t and $\bar{U}_t$ are randomly generated, and then fixed in all runs of the entire numerical experiment. The input u_t at time $0 \le t \le \ell - 1$ in each rollout is generated from Gaussian distribution $\mathcal{N}(\nu_t, \bar{U}_t)$. In addition, ℓ is set to be 4. Since RLS is based on single trajectory data, we set the length of the trajectory as ℓn_r, so that the number of samples that RLS uses is the same as that of Algorithm 1. We consider RLS with independent standard Gaussian inputs as a baseline. In order to rule out the effect of different input design, we also consider RLS with periodic inputs (RLSp for short) satisfying that in each period, the inputs are generated in the same way as those in a rollout of Algorithm 1.

For each system, we run the three algorithms, RLS, RLSp, and Algorithm 1, for 50 times respectively. The mean of estimation error in each case is presented in Fig. 5. It can be observed that RLS and RLSp perform similarly for all cases. When multiplicative noise is absent, they converge slightly quicker than Algorithm 1. They are also a little better than Algorithm 1, in the case $\rho(A) = 0.6$ with noise, for the estimation of $[A\ B]$, indicating OLS could be applied in Algorithm 1 to replace the averaging step. However, Algorithm 1 surpasses RLS and RLSp in identifying the noise covariance matrix. Moreover the performance of RLS gets worse as $\rho(A)$ grows. Interestingly, in the case of $\rho(A) = 0.8$, although the nominal system is stable, the second-moment dynamic of system states is not. This leads to degraded performance of RLS when estimating $[A\ B]$ and divergence of RLS estimating the covariance matrix. In the marginally stable case, namely $\rho(A) = 1$, RLS and RLSp explode in finite time. In contrast, Algorithm 1 behaves almost identically for all cases (the consistency of Algorithm 1 in the marginally stable case is shown in Fig. 1). To sum up, Algorithm 1 can deal with the estimation of noise covariance matrix better and relies less on the stability of both the nominal system and the second-moment dynamic of system states.

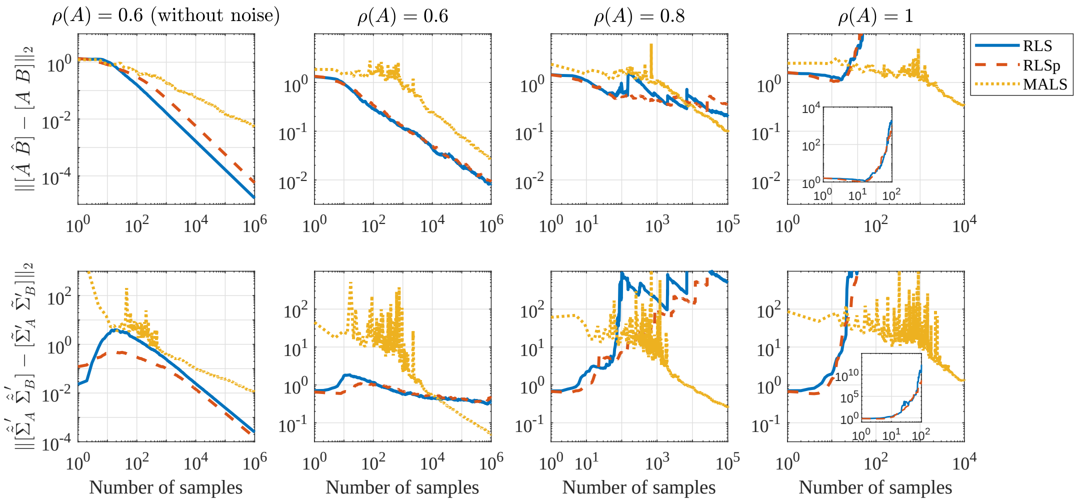

Fig. 5. Performance comparison of RLS, RLSp, and Algorithm 1.

5 Conclusion and Future Work

In this paper we proposed an identification algorithm for linear systems with multiplicative noise based on multiple trajectory data. By designing appropriate exciting inputs, the proposed algorithm is able to jointly estimate the nominal system and multiplicative noise covariance. The asymptotic and non-asymptotic performance of the algorithm were analyzed, and illustrated by numerical simulations. Future work include studying more efficient algorithms that can be used in online setting, optimal and adaptive design of inputs, sparsity-promoting regularization for identification of networked systems, and obtaining end-to-end finite-sample performance guarantees for identification-based optimal control.

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A Proof of Proposition 3

We begin with a standard result from real analysis [40, 41] regarding the zero set of a polynomial.

Lemma 2 *A polynomial function $\mathbb{R}^n$ to $\mathbb{R}$ is either identically 0 or non-zero almost everywhere.*

We only need to consider the case with $\ell = n + m$, implying $\mathbf{Z}$ is a square matrix. Note that when (A, B) and μ_0 are fixed, $|\mathbf{Z}|$ is a polynomial of $\nu := [\nu_0^\intercal \ \cdots \ \nu_{\ell-1}^\intercal]^\intercal \in \mathbb{R}^{m\ell}$. Hence if we can find a vector in $\mathbb{R}^{m\ell}$ such that $|\mathbf{Z}| \neq 0$, then we know that $|\mathbf{Z}| \neq 0$ almost everywhere from Lemma 2.

First we verify the conclusion for $\mu_0 = \mathbf{0}_n$. It follows from the definition of controllability and the assumption that $[A^{n-1}B \ A^{n-2}B \ \cdots \ B]$ has full row rank. Without loss of generality, let B_1, AB_1, $\ldots$, $A^{r_1}B_1$, B_2, $\ldots$, $A^{r_2}B_2$, $\ldots$, B_p, $\ldots$, $A^{r_p}B_p$ be a basis of $\mathbb{R}^n$, where B_i is the i-th column of B, $1 \le p \le m$, $0 \le r_i \le n-1$, $1 \le i \le p$, and $p + \sum_{i=1}^p r_i = n$. Moreover, A^kB_i, $k > r_i$ and $1 \le i \le p$, can be written as a linear combination of B_1, $\ldots$, $A^{r_1}B_1$, $\cdots$, $A^{r_i}B_i$.

Let $\tilde{\nu} := [\tilde{\nu}_0^\intercal \ \cdots \ \tilde{\nu}_{n-1}^\intercal]^\intercal$ be such that $\tilde{\nu}_0 = \mathbf{e}_1^m$, $\tilde{\nu}_{q_1} = \mathbf{e}_2^m$, $\ldots$, $\tilde{\nu}_{q_k} = \mathbf{e}_{k+1}^m$, $\ldots$, $\tilde{\nu}_{q_{p-1}} = \mathbf{e}_p^m$ (or any nonzero multiplier of respective unit vectors), where $q_k = k + \sum_{i=1}^k r_i$, $1 \le k \le p$, and $\tilde{\nu}_i = \mathbf{0}_m$ for other $1 \le i \le n-1$. Then for $\tilde{\mu}_{t+1} = A\tilde{\mu}_t + B\tilde{\nu}_t$, $0 \le t \le n-1$, and $\tilde{\mu}_0 = \mathbf{0}_n$, it holds that

$$\begin{aligned}
&\tilde{\mu}_t = A^{t-1}B_1, && 1 \le t \le q_1,\\
&\tilde{\mu}_t = A^{t-q_1-1}B_2 + A^{t-1}B_1, && q_1 + 1 \le t \le q_1 + q_2,\\
&\vdots && \vdots\\
&\tilde{\mu}_t = A^{t-q_{p-1}-1}B_p + \cdots + A^{t-q_1-1}B_2 + A^{t-1}B_1, && q_{p-1} + 1 \le t \le n.
\end{aligned}$$

From the definition of B_1, $\ldots$, $A^{r_p}B_p$, and q_k, we know that $\tilde{\mu}_1$, $\ldots$, $\tilde{\mu}_n$ are linearly independent. Hence we show that there exists a vector $[\nu_0^\intercal \ \cdots \ \nu_{n-1}^\intercal]^\intercal$ such that $[\mu_1 \ \cdots \ \mu_n]$ has full rank. If $m = 1$, then let $\nu = [\tilde{\nu}^\intercal \ 0]^\intercal$ and

$$\mathbf{Z} = \begin{bmatrix} \tilde{\mu}_n & \cdots & \tilde{\mu}_1 & \mathbf{0}_n \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

has full rank. In the case of $m \ge 2$, fix ν_n to be zero. Set $\nu_{n+1} = c_1\mathbf{e}_2^m$, $c_1 \in \mathbb{R}$, and

$$\begin{vmatrix} \mu_{n+1} & \tilde{\mu}_n & \cdots & \cdots & \tilde{\mu}_{q_1} & \cdots & \cdots & \mathbf{0}_n \\ 0 & 0 & \cdots & \cdots & 0 & \cdots & 0 & 1 \\ c_1 & 0 & \cdots & 0 & d_1 & 0 & \cdots & 0 \end{vmatrix} = c_1 \begin{vmatrix} \tilde{\mu}_n & \cdots & \tilde{\mu}_1 & \mathbf{0}_n \\ 0 & \cdots & 0 & 1 \end{vmatrix} + (-1)^{2n+4-q_1} d_1 \begin{vmatrix} \mu_{n+1} & \tilde{\mu}_n & \cdots & \tilde{\mu}_{q_1+1} & \tilde{\mu}_{q_1-1} & \cdots & \mathbf{0}_n \\ 0 & 0 & \cdots & \cdots & \cdots & 0 & 1 \end{vmatrix},$$

where $d_1 = 1$ if $p > 1$ and $d_1 = 0$ if $p = 1$. Hence there must exist $\tilde{c}_1 \in \mathbb{R}$, such that the above determinant is nonzero. Inductively, set $\nu_{n+k} = c_k\mathbf{e}_{k+1}^m$, $2 \le k \le m-1$, and we can find $\tilde{c}_k$, $2 \le k \le m-1$, (consequently $\tilde{\nu}_0$, $\ldots$, $\tilde{\nu}_{n+m-1}$) such that $|\mathbf{Z}| \neq 0$.

Now suppose that the system starts with $\mu_0 \neq \mathbf{0}_n$. Since $[A^{n-1}B \ A^{n-2}B \ \cdots \ B]$ has full row rank, so does the matrix $[A^{n-1}B \ A^{n-2}B \ \cdots \ B \ A^{n-1}\mu_0 \ \cdots \ \mu_0]$. Hence without loss of generality we can assume that μ_0, $A\mu_0$, $\ldots$, $A^{r_0}\mu_0$, B_1, $\ldots$, $A^{r_1}B_1$, $\ldots$, B_p, $\ldots$, $A^{r_p}B_p$ is a basis of $\mathbb{R}^n$, where $0 \le p \le m$ ($p = 0$ means there is no B_i), $0 \le r_i \le n-1$, $0 \le i \le p$, and $p + 1 + \sum_{i=0}^p r_i = n$. Moreover, $r_i \le n-2$ for all $1 \le i \le p$. One can verify that there exists a vector $[\nu_0^\intercal \ \cdots \ \nu_{n-1}^\intercal]^\intercal$ such that $[\mu_0 \ \cdots \ \mu_{n-1}]$ has full rank. Therefore in a similar way the conclusion can be obtained.

Remark 11 *From the proof we know that even if (A, B) is not controllable, $\mathbf{Z}$ can still have full row rank as long as $[A^{n-1}B \ A^{n-2}B \ \cdots \ B \ A^{n-1}\mu_0 \ \cdots \ \mu_0]$ has full row rank for some $\mu_0 \in \mathbb{R}^n$.*

B Proof of Proposition 4

Write (5) as

$$\begin{aligned}\tilde{X}_{t+1} &= (\tilde{A} + \tilde{\Sigma}'_A)\tilde{X}_t + (\tilde{B} + \tilde{\Sigma}'_B)\tilde{U}_t + [K_{BA}W_t + K_{AB}W'_t] \\ &:= \breve{A}\tilde{X}_t + \breve{B}\tilde{U}_t + \eta_t.\end{aligned}$$

Note that after setting $\nu_t = 0$ for all $0 \le t \le \ell - 1$, $\tilde{U}_t = \mathrm{svec}(\bar{U}_t)$ and the above dynamic becomes

$$\tilde{X}_{t+1} = \breve{A}\tilde{X}_t + \breve{B}\tilde{U}_t.$$

Similar to the proof of Proposition 3, we first examine if $\tilde{X}_0$, $\breve{A}\tilde{X}_0$, $\ldots$, $\breve{A}^{n(n+1)/2-1}\tilde{X}_0$ are linearly independent. If not, we can select some columns of $[\breve{A}^{n(n+1)/2-1}\breve{B}, \ldots, \breve{B}]$ to together form a basis of $\mathbb{R}^{n(n+1)/2}$. The rest is essentially the same as the proof of Proposition 3, by considering $\tilde{U}_t$ as an input.

C Proof of Theorem 1

Consider each rollout $[(x_0^{(k)})^\intercal, \ldots, (x_\ell^{(k)})^\intercal]^\intercal$ as an independent sample of the random vector $\mathbf{x}_\ell := [x_0^\intercal, \ldots, x_\ell^\intercal]^\intercal$, and from Assumption 1 (ii) and (iii) we know that the random vector $\mathbf{x}_\ell$ has finite first and second moments. So it follows from the Kolmogorov's strong law of large numbers that $\hat{\mathbf{Y}} \to \mathbf{Y}$ a.s., and similarly $\hat{\mathbf{Z}} \to \mathbf{Z}$ a.s., as $n_r \to \infty$. Hence $\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal \to \mathbf{Y}\mathbf{Z}^\intercal$ and $\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal \to \mathbf{Z}\mathbf{Z}^\intercal$ a.s. From the assumption that $\mathbf{Z}\mathbf{Z}^\intercal$ is invertible and the continuous mapping theorem (Theorem 2.3 of [42]), it can be obtained that as $n_r \to \infty$

$$(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1} \to (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}, \text{ a.s.}$$

When $(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}$ does not exist, in Algorithm 1 we replace it by $(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger$. Thus $(\hat{A}, \hat{B}) \to (A, B)$. Combining the above convergence with the Kolmogorov's strong law of large numbers, the convergence of $\hat{\mathbf{C}}$ and $\hat{\mathbf{D}}$ follows. Therefore, applying the continuous mapping theorem again, we obtain the consistency of the estimator $(\hat{\tilde{\Sigma}}'_A, \hat{\tilde{\Sigma}}'_B)$.

D Proof of Lemma 1

For the first claim, regarding states, taking the norm of both sides of System (1), at time step t we have

$$\big\|x_{t+1}^{(k)}\big\| = \big\|(A + \bar{A}_t^{(k)})x_t^{(k)} + (B + \bar{B}_t^{(k)})u_t^{(k)}\big\|. \tag{D.1}$$

Using the triangle inequality and the fact that the spectral norm is compatible with the Euclidean norm, we have

$$\begin{aligned}\big\|x_{t+1}^{(k)}\big\| &\le \big\|(A + \bar{A}_t^{(k)})x_t^{(k)}\big\| + \big\|(B + \bar{B}_t^{(k)})u_t^{(k)}\big\| \\ &\le \big\|A + \bar{A}_t^{(k)}\big\|_2\big\|x_t^{(k)}\big\| + \big\|B + \bar{B}_t^{(k)}\big\|_2\big\|u_t^{(k)}\big\|.\end{aligned}$$

Using the triangle inequality and Assumption 2 (iii) we have

$$\begin{aligned}\big\|A + \bar{A}_t^{(k)}\big\|_2 &\le \max_{0\le t\le \ell}\big\|A + \bar{A}_t^{(k)}\big\|_2 \le \|A\|_2 + \max_{0\le t\le \ell}\big\|\bar{A}_t^{(k)}\big\|_2 =: c_A, \\ \big\|B + \bar{B}_t^{(k)}\big\|_2 &\le \max_{0\le t\le \ell}\big\|B + \bar{B}_t^{(k)}\big\|_2 \le \|B\|_2 + \max_{0\le t\le \ell}\big\|\bar{B}_t^{(k)}\big\|_2 =: c_B,\end{aligned}$$

so from Assumption 2 (ii)

$$\big\|x_{t+1}^{(k)}\big\| \le c_A\big\|x_t^{(k)}\big\| + c_B c_U. \tag{D.2}$$

For the base case when $t = 0$, we have by Assumption 2 (i) that $\big\|x_0^{(k)}\big\| \le c_X$. Applying (D.2) inductively with the base case proves the first claim.

For the second claim, regarding the state-deviations, we have

$$\begin{aligned}e_{t+1}^{(k)} &= x_{t+1}^{(k)} - \mathbb{E}\{x_{t+1}^{(k)}\} \\ &= \big(A + \bar{A}_t^{(k)}\big)x_t^{(k)} + \big(B + \bar{B}_t^{(k)}\big)u_t^{(k)} - \mathbb{E}\big\{\big(A + \bar{A}_t^{(k)}\big)x_t^{(k)} + \big(B + \bar{B}_t^{(k)}\big)u_t^{(k)}\big\} \\ &= \big(A + \bar{A}_t^{(k)}\big)x_t^{(k)} + \big(B + \bar{B}_t^{(k)}\big)u_t^{(k)} - A\mathbb{E}\{x_t^{(k)}\} - B\mathbb{E}\{u_t^{(k)}\} \\ &= A\big(x_t^{(k)} - \mathbb{E}\{x_t^{(k)}\}\big) + B\big(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\}\big) + \bar{A}_t^{(k)}x_t^{(k)} + \bar{B}_t^{(k)}u_t^{(k)} \\ &= Ae_t^{(k)} + Bd_t^{(k)} + \bar{A}_t^{(k)}x_t^{(k)} + \bar{B}_t^{(k)}u_t^{(k)}.\end{aligned}$$

Taking the norm of both sides, using submultiplicativity and triangle inequality, we have

$$\begin{aligned}\|e_{t+1}^{(k)}\| &= \|Ae_t^{(k)} + Bd_t^{(k)} + \bar{A}_t^{(k)}x_t^{(k)} + \bar{B}_t^{(k)}u_t^{(k)}\| \\ &\le \|A\|_2\|e_t^{(k)}\| + \|B\|_2\|d_t^{(k)}\| + \|\bar{A}_t^{(k)}\|_2\|x_t^{(k)}\| + \|\bar{B}_t^{(k)}\|_2\|u_t^{(k)}\| \\ &\le \|A\|_2\|e_t^{(k)}\| + \|B\|_2 c_\nu + c_{\bar{A}}c_M + c_{\bar{B}}c_U,\end{aligned} \tag{D.3}$$

where the final inequality follows from the first part of Lemma 1 and Assumptions 2 (ii) and (iii). For the base case when $t = 0$, we have by Assumption 2 (i) that $\big\|e_0^{(k)}\big\| \le c_\mu$. Applying (D.3) inductively with the base case proves the second claim.

By the definition of the spectral norm and the first and second claims we have

$$\begin{aligned}\big\|x_t^{(k)}x_t^{(k)\mathsf{T}}\big\|_2 &= \big\|x_t^{(k)}\big\|_2^2 \le c_M^2, \\ \big\|e_t^{(k)}e_t^{(k)\mathsf{T}}\big\|_2 &= \big\|e_t^{(k)}\big\|_2^2 \le c_N^2,\end{aligned}$$

proving the third and fourth claims.

For the fifth and sixth claims, define the quantities

$$\begin{aligned}
\Delta X_t &:= \mathrm{vec}\left(x_t^{(k)}(x_t^{(k)})^\intercal - \mathbb{E}\{x_t^{(k)}(x_t^{(k)})^\intercal\}\right) = \mathrm{vec}\left(x_t^{(k)}(x_t^{(k)})^\intercal\right) - X_t,\\
\Delta U_t &:= \mathrm{vec}\left(u_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\{u_t^{(k)}(u_t^{(k)})^\intercal\}\right) = \mathrm{vec}\left(u_t^{(k)}(u_t^{(k)})^\intercal\right) - U_t,\\
\Delta W_t &:= \mathrm{vec}\left(x_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\{x_t^{(k)}(u_t^{(k)})^\intercal\}\right) = \mathrm{vec}\left(x_t^{(k)}(u_t^{(k)})^\intercal\right) - W_t.
\end{aligned}$$

We can bound $\|\Delta U_t\|$ as

$$\begin{aligned}
\|\Delta U_t\| &= \left\|\mathrm{vec}\left(u_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\{u_t^{(k)}(u_t^{(k)})^\intercal\}\right)\right\|\\
&= \left\|\mathrm{vec}\left(u_t^{(k)}(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})^\intercal + (u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})\mathbb{E}\{u_t^{(k)}\}^\intercal - \mathbb{E}\{u_t^{(k)}(u_t^{(k)} - \mathbb{E}u_t^{(k)})^\intercal\}\right)\right\|\\
&= \left\|u_t^{(k)}(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})^\intercal + (u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})\mathbb{E}\{u_t^{(k)}\}^\intercal - \mathbb{E}\{u_t^{(k)}(u_t^{(k)} - \mathbb{E}u_t^{(k)})^\intercal\}\right\|_F\\
&\le \left\|u_t^{(k)}(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})^\intercal\right\|_F + \left\|(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})\mathbb{E}\{u_t^{(k)}\}^\intercal\right\|_F + \mathbb{E}\left\{\left\|u_t^{(k)}(u_t^{(k)} - \mathbb{E}u_t^{(k)})^\intercal\right\|_F\right\}\\
&\le 3c_U c_\nu.
\end{aligned}$$

For the sixth claim, we can bound $\|\Delta W_t\|$ as

$$\begin{aligned}
\|\Delta W_t\| &= \left\|\mathrm{vec}\left(x_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\{x_t^{(k)}(u_t^{(k)})^\intercal\}\right)\right\|\\
&= \left\|\mathrm{vec}\left((x_t^{(k)} - \mathbb{E}\{x_t^{(k)}\})(u_t^{(k)})^\intercal + \mathbb{E}\{x_t^{(k)}\}(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})^\intercal\right)\right\|\\
&= \left\|(x_t^{(k)} - \mathbb{E}\{x_t^{(k)}\})(u_t^{(k)})^\intercal + \mathbb{E}\{x_t^{(k)}\}(u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\})^\intercal\right\|_F\\
&\le \|x_t^{(k)} - \mathbb{E}\{x_t^{(k)}\}\|\|u_t^{(k)}\| + \|\mathbb{E}\{x_t^{(k)}\}\|\|u_t^{(k)} - \mathbb{E}\{u_t^{(k)}\}\|\\
&\le c_N c_U + c_M c_\nu =: c_W.
\end{aligned}$$

For the fifth claim, substituting the dynamics and expanding the products we have

$$\begin{aligned}
\Delta X_{t+1} &= \mathrm{vec}\left(x_{t+1}^{(k)}(x_{t+1}^{(k)})^\intercal - \mathbb{E}\{x_{t+1}^{(k)}(x_{t+1}^{(k)})^\intercal\}\right)\\
&= \mathrm{vec}\Big(\left[(A+\bar{A}_t^{(k)})x_t^{(k)} + (B+\bar{B}_t^{(k)})u_t^{(k)}\right]\left[(A+\bar{A}_t^{(k)})x_t^{(k)} + (B+\bar{B}_t^{(k)})u_t^{(k)}\right]^\intercal\\
&\qquad -\mathbb{E}\left\{\left[(A+\bar{A}_t^{(k)})x_t^{(k)} + (B+\bar{B}_t^{(k)})u_t^{(k)}\right]\left[(A+\bar{A}_t^{(k)})x_t^{(k)} + (B+\bar{B}_t^{(k)})u_t^{(k)}\right]^\intercal\right\}\Big)\\
&= \mathrm{vec}\Bigg((A+\bar{A}_t^{(k)})x_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal + (A+\bar{A}_t^{(k)})x_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\\
&\qquad + (B+\bar{B}_t^{(k)})u_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal + (B+\bar{B}_t^{(k)})u_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\\
&\quad - \mathbb{E}\{(A+\bar{A}_t^{(k)})x_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal + (A+\bar{A}_t^{(k)})x_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\\
&\qquad + (B+\bar{B}_t^{(k)})u_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal + (B+\bar{B}_t^{(k)})u_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\}\Bigg)\\
&= \mathrm{vec}\Bigg((A+\bar{A}_t^{(k)})x_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal - \mathbb{E}\{(A+\bar{A}_t^{(k)})x_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal\}\\
&\quad (A+\bar{A}_t^{(k)})x_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal - \mathbb{E}\{(A+\bar{A}_t^{(k)})x_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\}\\
&\quad (B+\bar{B}_t^{(k)})u_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal - \mathbb{E}\{(B+\bar{B}_t^{(k)})u_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal\}\\
&\quad (B+\bar{B}_t^{(k)})u_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal - \mathbb{E}\{(B+\bar{B}_t^{(k)})u_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\}\Bigg).
\end{aligned} \tag{D.4}$$

Considering the first of the four terms of (D.4), we have

$$
\begin{aligned}
\boxed{1} &:= \operatorname{vec}\Big((A+\bar{A}_t^{(k)})x_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal - \mathbb{E}\big\{(A+\bar{A}_t^{(k)})x_t^{(k)}(x_t^{(k)})^\intercal(A+\bar{A}_t^{(k)})^\intercal\big\}\Big) \\
&= \operatorname{vec}\Bigg(Ax_t^{(k)}(x_t^{(k)})^\intercal A^\intercal + Ax_t^{(k)}(x_t^{(k)})^\intercal(\bar{A}_t^{(k)})^\intercal + \bar{A}_t^{(k)}x_t^{(k)}(x_t^{(k)})^\intercal A^\intercal + \bar{A}_t^{(k)}x_t^{(k)}(x_t^{(k)})^\intercal(\bar{A}_t^{(k)})^\intercal \\
&- A\mathbb{E}\big\{x_t^{(k)}(x_t^{(k)})^\intercal\big\}A^\intercal - 0 - 0 - \mathbb{E}\big\{\bar{A}_t^{(k)}x_t^{(k)}(x_t^{(k)})^\intercal(\bar{A}_t^{(k)})^\intercal\big\}\Bigg) \\
&= (A\otimes A)\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal - \mathbb{E}\big\{x_t^{(k)}(x_t^{(k)})^\intercal\big\}\Big) \\
&\quad + (A\otimes \bar{A}_t^{(k)})\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big) + (\bar{A}_t^{(k)}\otimes A)\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big) \\
&\quad + (\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)})\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big) - \mathbb{E}\big\{(\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)})\big\}\mathbb{E}\Big\{\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big)\Big\} \\
&= (A\otimes A)\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal - \mathbb{E}\big\{x_t^{(k)}(x_t^{(k)})^\intercal\big\}\Big) \\
&\quad + (A\otimes \bar{A}_t^{(k)})\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big) + (\bar{A}_t^{(k)}\otimes A)\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big) \\
&\quad + \Big[(\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)}) - \mathbb{E}\big\{(\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)})\big\}\Big]\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal\Big) \\
&\quad + \mathbb{E}\big\{(\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)})\big\}\operatorname{vec}\Big(x_t^{(k)}(x_t^{(k)})^\intercal - \mathbb{E}\big\{x_t^{(k)}(x_t^{(k)})^\intercal\big\}\Big).
\end{aligned}
$$

Taking norms, and substituting notated quantities, we have

$$
\begin{aligned}
\Big\|\boxed{1}\Big\| &\le \|A\otimes A\|_2\|\Delta X_t\| + \|A\otimes \bar{A}_t^{(k)}\|_2\|\operatorname{vec}(x_t^{(k)}(x_t^{(k)})^\intercal)\| + \|\bar{A}_t^{(k)}\otimes A\|_2\|\operatorname{vec}(x_t^{(k)}(x_t^{(k)})^\intercal)\| \\
&\quad + \|(\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)}) - \Sigma'_A\|_2\|\operatorname{vec}(x_t^{(k)}(x_t^{(k)})^\intercal)\| + \|\Sigma'_A\|_2\|\Delta X_t\| \\
&\le (\|A\|_2^2 + \|\Sigma'_A\|_2)\|\Delta X_t\| + \Big(2\|A\|_2\|\bar{A}_t^{(k)}\|_2 + \|(\bar{A}_t^{(k)}\otimes \bar{A}_t^{(k)}) - \Sigma'_A\|_2\Big)\|x_t^{(k)}\|^2 \\
&\le (\|A\|_2^2 + \|\Sigma'_A\|_2)\|\Delta X_t\| + (2\|A\|_2 c_{\bar{A}} + c_{\Sigma'_A})c_M^2.
\end{aligned}
$$

Applying identical arguments to the fourth term of (D.4)

$$
\boxed{4} := \operatorname{vec}\Big((B+\bar{B}_t^{(k)})u_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal - \mathbb{E}\big\{(B+\bar{B}_t^{(k)})u_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\big\}\Big),
$$

we obtain the norm bound

$$
\begin{aligned}
\Big\|\boxed{4}\Big\| &\le (\|B\|_2^2 + \|\Sigma'_B\|_2)\|\Delta U_t\| + (2\|B\|_2 c_{\bar{B}} + c_{\Sigma'_B})c_U^2 \\
&\le 3(\|B\|_2^2 + \|\Sigma'_B\|_2)c_U c_\nu + (2\|B\|_2 c_{\bar{B}} + c_{\Sigma'_B})c_U^2.
\end{aligned}
$$

Likewise, for the second term of (D.4), we have

$$
\begin{aligned}
\boxed{2} &:= \operatorname{vec}\Big((A+\bar{A}_t^{(k)})x_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal - \mathbb{E}\big\{(A+\bar{A}_t^{(k)})x_t^{(k)}(u_t^{(k)})^\intercal(B+\bar{B}_t^{(k)})^\intercal\big\}\Big) \\
&= \operatorname{vec}\Bigg(Ax_t^{(k)}(u_t^{(k)})^\intercal B^\intercal + Ax_t^{(k)}(u_t^{(k)})^\intercal(\bar{B}_t^{(k)})^\intercal + \bar{A}_t^{(k)}x_t^{(k)}(u_t^{(k)})^\intercal B^\intercal + \bar{A}_t^{(k)}x_t^{(k)}(u_t^{(k)})^\intercal(\bar{B}_t^{(k)})^\intercal \\
&- A\mathbb{E}\big\{x_t^{(k)}(u_t^{(k)})^\intercal\big\}B^\intercal - 0 - 0 - \mathbb{E}\big\{\bar{A}_t^{(k)}x_t^{(k)}(u_t^{(k)})^\intercal(\bar{B}_t^{(k)})^\intercal\big\}\Bigg) \\
&= (B\otimes A)\operatorname{vec}\Big(x_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\big\{x_t^{(k)}(u_t^{(k)})^\intercal\big\}\Big) \\
&\quad + (B\otimes \bar{A}_t^{(k)})\operatorname{vec}\Big(x_t^{(k)}(u_t^{(k)})^\intercal\Big) + (\bar{B}_t^{(k)}\otimes A)\operatorname{vec}\Big(x_t^{(k)}(u_t^{(k)})^\intercal\Big)
\end{aligned}
$$

$$
\begin{aligned}
&+ (\bar{B}_t^{(k)} \otimes \bar{A}_t^{(k)}) \operatorname{vec}\left(x_t^{(k)} (u_t^{(k)})^\intercal\right) - \mathbb{E}\big\{(\bar{B}_t^{(k)} \otimes \bar{A}_t^{(k)})\big\}\mathbb{E}\big\{ \operatorname{vec}\left(x_t^{(k)} (u_t^{(k)})^\intercal\right) \big\} \\
= {}& (B \otimes A) \operatorname{vec}\left(x_t^{(k)} (u_t^{(k)})^\intercal - \mathbb{E}\{x_t^{(k)} (u_t^{(k)})^\intercal\}\right) \\
&+ (B \otimes \bar{A}_t^{(k)}) \operatorname{vec}\left(x_t^{(k)} (u_t^{(k)})^\intercal\right) + (\bar{B}_t^{(k)} \otimes A) \operatorname{vec}\left(x_t^{(k)} (u_t^{(k)})^\intercal\right) \\
&+ (\bar{B}_t^{(k)} \otimes \bar{A}_t^{(k)}) \operatorname{vec}\left(x_t^{(k)} (u_t^{(k)})^\intercal\right).
\end{aligned}
$$

Taking norms, and substituting notated quantities, we have

$$
\left\| \boxed{2} \right\| \leq \|A\|_2 \|B\|_2 \|\Delta W_t\| + (\|A\|_2 c_{\bar{B}} + \|B\|_2 c_{\bar{A}} + c_{\bar{A}} c_{\bar{B}}) c_M c_U.
$$

The third term of (D.4) is simply the transpose of the second term, so an identical norm bound holds.

Putting together the four terms of (D.4), we obtain

$$
\begin{aligned}
\|\Delta X_{t+1}\| \leq {}& (\|A\|_2^2 + \|\Sigma'_A\|_2)\|\Delta X_t\| + (2\|A\|_2 c_{\bar{A}} + c_{\Sigma'_A}) c_M^2 \\
&+ 3(\|B\|_2^2 + \|\Sigma'_B\|_2) c_U c_\nu + (2\|B\|_2 c_{\bar{B}} + c_{\Sigma'_B}) c_U^2 \\
&+ 2\|A\|_2 \|B\|_2 c_W + (2\|A\|_2 c_{\bar{B}} + 2\|B\|_2 c_{\bar{A}} + 2 c_{\bar{A}} c_{\bar{B}}) c_M c_U.
\end{aligned}
\tag{D.5}
$$

For the base case when $t = 0$, we have by Assumption 2 (v) that $\|\Delta X_0\| \leq c_{\Delta X}$. Applying (D.5) inductively with the base case proves the fifth claim.

E Basic identities and inequalities

In the proofs of Theorems 2 and 3, the following facts will be used.

Submultiplicativity of spectral norm For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$,

$$\|AB\|_2 \leq \|A\|_2 \|B\|_2.$$

Norm of Kronecker product For $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$,

$$\|A \otimes B\|_2 = \|A\|_2 \|B\|_2.$$

Inverse of spectral norm For any invertible matrix $A \in \mathbb{R}^{n\times n}$ we have

$$\|A^{-1}\|_2 = \frac{1}{\sqrt{\lambda_{\min}(AA^\intercal)}}. \tag{E.1}$$

Difference of matrix inverses Suppose $A, E \in \mathbb{R}^{n\times n}$ are invertible square matrices. Then

$$A^{-1} - E^{-1} = E^{-1}(((E - A) + A)A^{-1}) - E^{-1} = E^{-1}(E - A)A^{-1}. \tag{E.2}$$

Matrix inverse perturbation bound (Equation (5.8.1) of [37])
Suppose $A, A + F \in \mathbb{R}^{n\times n}$ are invertible square matrices. Then

$$\|A^{-1} - (A + F)^{-1}\| \leq \|A^{-1}\| \cdot \|F\| \cdot \|(A + F)^{-1}\|. \tag{E.3}$$

This follows from taking $E = A + F$ in (E.2) and using submultiplicativity of spectral norm.

Probability bound on the sum of random variables
Consider k random variables, $X_1, \ldots, X_k$, and a positive number ε. Note that $X_i < \varepsilon/k$ for all $i \in [k]$, implies $\sum_{i=1}^k X_i < \varepsilon$, so it follows from the union bound that

$$\mathbb{P}\left\{\sum_{i=1}^k X_i \geq \varepsilon\right\} \leq \sum_{i=1}^k \mathbb{P}\{X_i \geq \varepsilon/k\}. \tag{E.4}$$

Probability bound on the product of nonnegative random variables
Consider two nonnegative random variables, X_1 and X_2, and a positive number ε. Since $X_1 < \sqrt{\varepsilon}$ and $X_2 < \sqrt{\varepsilon}$ implies $X_1X_2 < \varepsilon$ we have

$$\mathbb{P}\{X_1X_2 \geq \varepsilon\} \leq \mathbb{P}\{X_1 \geq \sqrt{\varepsilon}\} + \mathbb{P}\{X_2 \geq \sqrt{\varepsilon}\}. \tag{E.5}$$

We will need the following geometrical result later in the use of covering arguments.

Lemma 3 (Covering numbers of the Euclidean Sphere) *Consider a minimal γ-net $\{w_k, k \in [M_\gamma]\}$ of the n-dimensional sphere surface $\mathcal{S}_{n-1} := \{w \in \mathbb{R}^n : \|w\| = 1\}$. That is, for all $w \in \mathcal{S}_{n-1}$ there exists $w_i \in \{w_k, k \in [M_\gamma]\}$ such that $\|w - w_i\| \leq \gamma$, and M_γ is the smallest number satisfies this condition. Then for any $\gamma > 0$, the covering number M_γ, i.e., the cardinality of the γ-net satisfies*

$$\left(\frac{1}{\gamma}\right)^n \leq M_\gamma \leq \left(\frac{2}{\gamma} + 1\right)^n. \tag{E.6}$$

PROOF. A standard volume comparison involving Euclidean balls, e.g. Corollary 4.2.13 of [43], yields the result.

We also need the following matrix concentration inequality.

Lemma 4 (Matrix Bernstein inequality [44]) *Consider a finite sequence of independent random matrices* $\{X_k, k \in [N]\}$ *with common dimension* $m \times n$. *Assume that* $\mathbb{E}\{X_k\} = 0$ *and* $\|X_k\|_2 \leq L$, $k \in [N]$. *Introduce* $S := \sum_{k=1}^{N} X_k$ *and let* $v \geq \max\{\|\mathbb{E}\{SS^\intercal\}\|_2, \|\mathbb{E}\{S^\intercal S\}\|_2\}$. *Then, for all* $\varepsilon \geq 0$,

$$\mathbb{P}\{\|S\|_2 \geq \varepsilon\} \leq (n+m) \exp\left\{-\frac{3}{2} \cdot \frac{\varepsilon^2}{3v + L\varepsilon}\right\}. \tag{E.7}$$

We obtain the following corollary from Lemma 4.

Corollary 2 *Consider a finite sequence of independent random matrices* $\{Y_k, k \in [N]\}$ *with common dimension* $m \times n$. *Assume that* $\mathbb{E}\{Y_k\} = 0$ *and* $\|Y_k\|_2 \leq M$, $k \in [N]$. *Then, for all* $\varepsilon \geq 0$,

$$\mathbb{P}\left\{\left\|\frac{1}{N}\sum_{k=1}^{N} Y_k\right\|_2 \geq \varepsilon\right\} \leq \delta(\varepsilon),$$

where

$$\delta(\varepsilon) := (n+m) \exp\left\{-\frac{3}{2} \cdot \frac{N\varepsilon^2}{3M^2 + M\varepsilon}\right\}.$$

PROOF. Towards application of Lemma 4, assign $X_k = Y_k/N$ and thus $L = M/N$. Now we get a crude bound on v as

$$\begin{aligned}
\|\mathbb{E}\{SS^\intercal\}\|_2 &= \left\|\mathbb{E}\left\{\left(\sum_{k=1}^{N} X_k\right)\left(\sum_{j=1}^{N} X_j^\intercal\right)\right\}\right\|_2 \\
&= \left\|\sum_{k=1}^{N}\sum_{j=1}^{N} \mathbb{E}\left\{X_k X_j^\intercal\right\}\right\|_2 && \text{(linearity of } \mathbb{E}\{\cdot\}) \\
&= \left\|\sum_{k=1}^{N} \mathbb{E}\left\{X_k X_k^\intercal\right\}\right\|_2 && \text{(mutual independence of } X_k) \\
&\leq \sum_{k=1}^{N} \|\mathbb{E}\left\{X_k X_k^\intercal\right\}\|_2 && \text{(triangle inequality)} \\
&\leq \sum_{k=1}^{N} \mathbb{E}\left\{\|X_k X_k^\intercal\|_2\right\} && \text{(Jensen's inequality)} \\
&= \sum_{k=1}^{N} \mathbb{E}\left\{\|X_k\|_2^2\right\} && \text{(property of the spectral norm)} \\
&\leq \sum_{k=1}^{N} (M/N)^2 = M^2/N.
\end{aligned}$$

An identical argument shows $\|\mathbb{E}\{SS^\intercal\}\|_2 \leq M^2/N$ so we can take $v = M^2/N$. Applying Lemma 4 with $X_k = Y_k/N$, $L = M/N$, and $v = M^2/N$ yields the claim.

F Proof of Theorem 2

In this section we obtain bounds for system parameter error matrix $[\hat{A}\ \hat{B}] - [A\ B]$ by decomposing the difference using their representation in the least-squares estimators as

$$
\begin{aligned}
\big[\hat{A}\ \hat{B}\big] - [A\ B] &= \hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - \mathbf{Y}\mathbf{Z}^\intercal(\mathbf{Z}\mathbf{Z}^\intercal)^{-1} \\
&= \big[\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big](\mathbf{Z}\mathbf{Z}^\intercal)^{-1} + \mathbf{Y}\mathbf{Z}^\intercal\big[(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big] + \big[\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big]\big[(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big].
\end{aligned}
$$

In this form it is obvious that there are four unique terms, which fall into two groups. The first group is $\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal$ and $(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}$, which represent error terms amenable to analysis. The second group is $(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}$ and $\mathbf{Y}\mathbf{Z}^\intercal$, which are inherent to the system and do not depend on the estimator quality. The terms $\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal$ and $(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}$ are treated first, then the bound on $[\hat{A}\ \hat{B}] - [A\ B]$ is obtained.

Throughout this section, small probability bounds are denoted by $\delta_{[\cdot]}$, where $[\cdot]$ are various subscripts, and each of these bounds decreases monotonically towards 0 with increasing number of rollouts n_r.

The following bounds for $\hat{\mathbf{Y}} - \mathbf{Y}$ and $\hat{\mathbf{Z}} - \mathbf{Z}$ follow from Corollary 2.

Lemma 5 *Suppose that Assumptions 1 and 2 hold. Then for all $\varepsilon > 0$,*

$$
\mathbb{P}\big\{\big\|\hat{\mathbf{Y}} - \mathbf{Y}\big\|_2 \geq \varepsilon\big\} = \mathbb{P}\big\{\big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2 \geq \varepsilon\big\} = \mathbb{P}\Big\{\big[\big\|\hat{\mathbf{Y}} - \mathbf{Y}\big\|_2 \geq \varepsilon\big] \bigcup \big[\big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2 \geq \varepsilon\big]\Big\} \leq \delta_Y(\varepsilon),
$$

where

$$
\delta_Y(\varepsilon) := (n + \ell)\exp\left\{-\frac{3}{2}\cdot\frac{n_r\varepsilon^2}{3\ell c_N^2 + \varepsilon\sqrt{\ell c_N^2}}\right\},
$$

PROOF. Using the bound $\|e_t^{(k)}\| = \|x_t^{(k)} - \mathbb{E}\{x_t^{(k)}\}\| \leq c_N$ from Lemma 1, and denoting $\hat{\mathbf{Y}}_k := \big[x_\ell^{(k)}\ \cdots\ x_1^{(k)}\big]$ so $\hat{\mathbf{Y}} = (\sum_{k=1}^{n_r}\hat{\mathbf{Y}}_k)/n_r$ and $\mathbb{E}\{\hat{\mathbf{Y}}_k\} = \mathbf{Y}$, we obtain

$$
\big\|\hat{\mathbf{Y}}_k - \mathbf{Y}\big\|_2 \leq \big\|\hat{\mathbf{Y}}_k - \mathbf{Y}\big\|_F = \sqrt{\sum_{t=1}^{\ell}\big\|x_t^{(k)} - \mathbb{E}\{x_t^{(k)}\}\big\|^2} \leq \sqrt{\ell c_N^2}.
$$

Applying Corollary 2 with $Y_k = \hat{\mathbf{Y}}_k - \mathbf{Y}$, $N = n_r$, and $M = \sqrt{\ell c_N^2}$, we conclude

$$
\mathbb{P}\{\|\hat{\mathbf{Y}} - \mathbf{Y}\|_2 \geq \varepsilon\} \leq \delta_Y(\varepsilon).
$$

Denote $\hat{\mathbf{Z}}_k = \begin{bmatrix} x_{\ell-1}^{(k)} & \cdots & x_0^{(k)} \\ \nu_{\ell-1} & \cdots & \nu_0 \end{bmatrix}$ so $\hat{\mathbf{Z}} = (\sum_{k=1}^{n_r}\hat{\mathbf{Z}}_k)/n_r$ and $\mathbb{E}\{\hat{\mathbf{Z}}_k\} = \mathbf{Z}$. Noticing that the last m rows of $\hat{\mathbf{Z}}_k - \mathbf{Z}$ all have zero entries, we have that

$$
\big\|\hat{\mathbf{Y}} - \mathbf{Y}\big\|_2 = \big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2.
$$

Hence the events $\{\|\hat{\mathbf{Y}} - \mathbf{Y}\|_2 \geq \varepsilon\}$ and $\{\|\hat{\mathbf{Z}} - \mathbf{Z}\|_2 \geq \varepsilon\}$ are precisely the same, concluding the proof.

Remark 12 *The reason that the last m rows of $\hat{\mathbf{Z}}_k - \mathbf{Z}$ all have zero entries is that the first moments of the inputs are known in Algorithm 1, and appear identically in both $\hat{\mathbf{Z}}_k$ and $\mathbf{Z}$. Hence the probability bounds are independent of the input dimension m.*

Lemma 6 *Suppose that Assumptions 1 and 2 hold. Then for all* $\varepsilon > 0$*,*

$$\mathbb{P}\{\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\|_2 \geq \varepsilon\} \leq \delta_{YZ}(\varepsilon)$$

where

$$\delta_{YZ}(\varepsilon) := \delta_Y \left(\sqrt{\varepsilon + \left(\frac{\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2}{2}\right)^2} - \frac{\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2}{2} \right).$$

PROOF. We begin with the decomposition

$$\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal = (\hat{\mathbf{Y}} - \mathbf{Y})(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal + (\hat{\mathbf{Y}} - \mathbf{Y})\mathbf{Z}^\intercal + \mathbf{Y}(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal.$$

By the triangle inequality and submultiplicativity we have

$$\begin{aligned}
\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\|_2 &= \|(\hat{\mathbf{Y}} - \mathbf{Y})(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal + (\hat{\mathbf{Y}} - \mathbf{Y})\mathbf{Z}^\intercal + \mathbf{Y}(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal\|_2 \\
&\leq \|(\hat{\mathbf{Y}} - \mathbf{Y})(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal\|_2 + \|(\hat{\mathbf{Y}} - \mathbf{Y})\mathbf{Z}^\intercal\|_2 + \|\mathbf{Y}(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal\|_2 \\
&\leq \|\hat{\mathbf{Y}} - \mathbf{Y}\|_2\|\hat{\mathbf{Z}} - \mathbf{Z}\|_2 + \|\hat{\mathbf{Y}} - \mathbf{Y}\|_2\|\mathbf{Z}\|_2 + \|\mathbf{Y}\|_2\|\hat{\mathbf{Z}} - \mathbf{Z}\|_2 \\
&= \|\hat{\mathbf{Y}} - \mathbf{Y}\|_2^2 + \|\hat{\mathbf{Y}} - \mathbf{Y}\|_2 \left(\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2\right).
\end{aligned}$$

Considering a probability bound, solving the quadratic inequality in $\|\hat{\mathbf{Y}} - \mathbf{Y}\|_2$, and applying Lemma 5 we have

$$\begin{aligned}
\mathbb{P}\{\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\|_2 \geq \varepsilon\} &\leq \mathbb{P}\{\|\hat{\mathbf{Y}} - \mathbf{Y}\|_2^2 + \|\hat{\mathbf{Y}} - \mathbf{Y}\|_2(\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2) \geq \varepsilon\} \\
&= \mathbb{P}\left\{ \|\hat{\mathbf{Y}} - \mathbf{Y}\|_2 \geq \frac{1}{2}(\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2) \left(\sqrt{1 + \frac{4\varepsilon}{(\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2)^2}} - 1 \right) \right\} \\
&\leq \delta_Y \left(\frac{\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2}{2} \left(\sqrt{1 + \frac{4\varepsilon}{(\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2)^2}} - 1 \right) \right),
\end{aligned}$$

which was the claimed inequality.

Lemma 7 *Suppose Assumptions 1 and 2 hold. Given a positive constant* $\varepsilon_{\max}$*, then for all* $0 < \varepsilon < \varepsilon_{\max}$*,*

$$\mathbb{P}\{\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\|_2 \geq \varepsilon\} \leq \delta_{ZZ}(\varepsilon, \varepsilon_{\max}),$$

where

$$\begin{aligned}
\delta_{ZZ}(\varepsilon, \varepsilon_{\max}) &:= \delta_0 \left(\frac{1}{2}\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal) \left(1 - \frac{\varepsilon}{\varepsilon_{\max}}\right) \varepsilon \right) + \delta_m \left(\frac{\varepsilon \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal))} \right), \\
\delta_0(\varepsilon) &:= \delta_Y \left(\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal) + \varepsilon} - \sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)} \right), \\
\delta_m(\varepsilon) &:= \left(9^{n+m} + \left(\frac{16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)} + 1 \right)^{n+m} \right) \delta_0(\varepsilon).
\end{aligned}$$

Remark 13 *The additional parameter* $\varepsilon_{\max}$ *arises when bounding* $\lambda_{\min}^2(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)$*, and is in fact independent of the estimation bound.*

PROOF. Later we will show that $\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal$ is invertible with high probability, so we now assume the existence of $(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}$, i.e., $(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger = (\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}$. Hence we may apply (E.3) to obtain the decomposition

$$\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1} - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\|_2 = \|[(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal)\|_2$$

$$\leq \big\|(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}\big\|_2 \big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2.$$

Considering a probability bound, rearranging, and using (E.1) we obtain

$$\begin{aligned}
\mathbb{P}\big\{\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1} - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \geq \varepsilon\big\} &\leq \mathbb{P}\big\{\big\|(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}\big\|_2 \big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) - (\mathbf{Z}\mathbf{Z}^\intercal)\big\|_2 \geq \varepsilon\big\} \\
&= \mathbb{P}\left\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \frac{\varepsilon}{\big\|(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1}\big\|_2}\right\} \\
&= \mathbb{P}\big\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \varepsilon \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)\big\}. \qquad \text{(F.1)}
\end{aligned}$$

We are now faced with providing bounds on both $\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2$ on the left side and $\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)$ on the right side of the inequality inside the probability.

First, we consider the bound of $\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2$ by beginning with the decomposition

$$\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal = (\hat{\mathbf{Z}} - \mathbf{Z})(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal + (\hat{\mathbf{Z}} - \mathbf{Z})\mathbf{Z}^\intercal + \mathbf{Z}(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal.$$

Using the triangle inequality, submultiplicativity, and solving the quadratic inequality in $\big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2$, we obtain

$$\begin{aligned}
\mathbb{P}\big\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \varepsilon\big\} &= \mathbb{P}\big\{\big\|(\hat{\mathbf{Z}} - \mathbf{Z})(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal + (\hat{\mathbf{Z}} - \mathbf{Z})\mathbf{Z}^\intercal + \mathbf{Z}(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|(\hat{\mathbf{Z}} - \mathbf{Z})(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal\big\|_2 + \big\|(\hat{\mathbf{Z}} - \mathbf{Z})\mathbf{Z}^\intercal\big\|_2 + \big\|\mathbf{Z}(\hat{\mathbf{Z}} - \mathbf{Z})^\intercal\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2^2 + 2\big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2 \big\|\mathbf{Z}\big\|_2 \geq \varepsilon\big\} \\
&= \mathbb{P}\left\{\big\|\hat{\mathbf{Z}} - \mathbf{Z}\big\|_2 \geq \|\mathbf{Z}\|_2 \left(\sqrt{1 + \frac{\varepsilon}{\|\mathbf{Z}\|_2^2}} - 1\right)\right\}.
\end{aligned}$$

Applying Lemma 5 with the appropriate settings of ε yields

$$\mathbb{P}\big\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \varepsilon\big\} \leq \delta_Y\left(\|\mathbf{Z}\|_2 \left(\sqrt{1 + \frac{\varepsilon}{\|\mathbf{Z}\|_2^2}} - 1\right)\right) =: \delta_0(\varepsilon). \qquad \text{(F.2)}$$

Now we seek a lower bound of $\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)$. First an upper bound of $\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2$ is needed. To obtain this, we put forward a covering argument. Begin by constructing a quadratic form of $\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal$ with $w \in \mathbb{R}^{n+m}$, $\|w\| = 1$ and use the earlier result in (F.2) to obtain

$$\begin{aligned}
\mathbb{P}\big\{w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w > \|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\big\} &\leq \mathbb{P}\big\{w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w > w^\intercal \mathbf{Z}\mathbf{Z}^\intercal w + \varepsilon\big\} && \text{(definition of spectral norm)} \\
&= \mathbb{P}\big\{w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w - w^\intercal \mathbf{Z}\mathbf{Z}^\intercal w \geq \varepsilon\big\} \\
&= \mathbb{P}\big\{w^\intercal \big(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big) w \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \varepsilon\big\} && \text{(definition of spectral norm, } \|w\| = 1\text{)} \\
&\leq \delta_0(\varepsilon). && \text{(F.3)}
\end{aligned}$$

Consider a minimal γ-net $\{w_k, k \in [M_\gamma]\}$ of the $(n+m)$-sphere surface $\mathcal{S}_{n+m-1} := \{w \in \mathbb{R}^{n+m} : \|w\| = 1\}$. Hence for all $w \in \mathcal{S}_{n+m-1}$ there exists $k \in [M_\gamma]$ such that

$$\begin{aligned}
w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w &= (w - w_k)^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w + w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal (w - w_k) + w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k \\
&\leq 2\gamma \|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\|_2 + \max_{k \in [M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k,
\end{aligned}$$

where the inequality follows by $\|A\|_2 = \max_{\|x\|=1, \|y\|=1} y^\intercal A x$ for $A \in \mathbb{R}^n$ and $\|w - w_k\| \leq \gamma$. Taking supremum of the left side of the inequality over w, using the definition of the spectral norm, and rearranging implies that

$$\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2 \leq \frac{1}{1 - 2\gamma} \max_{k \in [M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k. \qquad \text{(F.4)}$$

By construction $\|w_k\| = 1$, so we may apply (F.3) to w_k in place of w from earlier in the proof:

$$\mathbb{P}\left\{w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k > \|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\right\} \leq \delta_0(\varepsilon).$$

Now apply the union bound over the M_γ terms to obtain

$$\begin{aligned}
\mathbb{P}\left\{\left[\max_{k\in[M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k\right] > \|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\right\} &= \mathbb{P}\left\{\bigcup_{k=1}^{M_\gamma}\left[w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k > \|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\right]\right\} \\
&\leq \sum_{k=1}^{M_\gamma} \mathbb{P}\big\{\big[w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k > \|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\big]\big\} \\
&\leq \sum_{k=1}^{M_\gamma} \delta_0(\varepsilon) = M_\gamma \delta_0(\varepsilon).
\end{aligned}$$

Note that tighter bounds can be obtained via more complicated arguments e.g. as in [45,46]. For definiteness, choose $\gamma = 1/4$. By (E.6) we know $M_\gamma \leq 9^{n+m}$. Thus (F.4) becomes

$$\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2 \leq 2 \max_{k\in[9^{n+m}]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k.$$

Considering a probability bound we have

$$\begin{aligned}
\mathbb{P}\big\{\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\|_2 \geq 2\left(\|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\right)\big\} &\leq \mathbb{P}\left\{2 \max_{k\in[9^{n+m}]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k \geq 2\left(\|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\right)\right\} \\
&= \mathbb{P}\left\{\max_{k\in[9^{n+m}]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k \geq \|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon\right\} \\
&\leq 9^{n+m}\delta_0(\varepsilon) =: \delta_1(\varepsilon). \qquad \text{(F.5)}
\end{aligned}$$

We are now in a position to derive a lower bound for $\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)$. Again we construct a quadratic form of $\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal$ with $w \in \mathbb{R}^{n+m}$, $\|w\| = 1$ and use the earlier result in (F.2) to obtain

$$\begin{aligned}
\mathbb{P}\big\{w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w < \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon\big\} &= \mathbb{P}\big\{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) > w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w + \varepsilon\big\} \\
&\leq \mathbb{P}\big\{w^\intercal \mathbf{Z}\mathbf{Z}^\intercal w > w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w + \varepsilon\big\} \qquad \text{(property of minimum eigenvalue)} \\
&= \mathbb{P}\big\{w^\intercal \mathbf{Z}\mathbf{Z}^\intercal w - w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w \geq \varepsilon\big\} \\
&= \mathbb{P}\big\{w^\intercal\big(\mathbf{Z}\mathbf{Z}^\intercal - \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big)w \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|\mathbf{Z}\mathbf{Z}^\intercal - \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2 \geq \varepsilon\big\} \qquad \text{(definition of spectral norm, } \|w\| = 1) \\
&\leq \delta_0(\varepsilon). \qquad \text{(F.6)}
\end{aligned}$$

Consider again a minimal γ-net $\{w_k, k \in [M_\gamma]\}$ of $\mathcal{S}_{n+m-1}$, and for all $w \in \mathcal{S}_{n+m-1}$, there exists $k \in [M_\gamma]$ such that

$$\begin{aligned}
w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w &= (w - w_k)^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w + w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal (w - w_k) + w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k \\
&\geq -2\gamma\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\|_2 + \min_{k\in[M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k, \qquad \text{(F.7)}
\end{aligned}$$

where the inequality follows by $\|A\|_2 = \max_{\|x\|=1,\|y\|=1} y^\intercal A x$ for $A \in \mathbb{R}^n$ and $\|w_k - w\| \leq \gamma$. By construction $\|w_k\| = 1$, so we may apply (F.6) to w_k in place of w from earlier in the proof:

$$\mathbb{P}\big\{w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k < \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon\big\} \leq \delta_0(\varepsilon).$$

As before, apply the union bound over the M_γ terms to obtain

$$\begin{aligned}
\mathbb{P}\left\{\left[\min_{k\in[M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k\right] < \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon\right\} &= \mathbb{P}\left\{\bigcup_{k=1}^{M_\gamma}\left[w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k < \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon\right]\right\} \\
&\leq \sum_{k=1}^{M_\gamma} \mathbb{P}\big\{\left[w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k < \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon\right]\big\} \\
&\leq \sum_{k=1}^{M_\gamma} \delta_0(\varepsilon) = M_\gamma \delta_0(\varepsilon).
\end{aligned}$$

For definiteness choose $\gamma = \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/(8\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)) = \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/(8\|\mathbf{Z}\mathbf{Z}^\intercal\|_2)$. By (E.6) we know

$$M_\gamma \leq \left(\frac{16\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)} + 1\right)^{n+m}.$$

Hence,

$$\mathbb{P}\left\{\left[\min_{k\in[M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k\right] < \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon\right\} \leq \left(\frac{16\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)} + 1\right)^{n+m} \delta_0(\varepsilon) =: \delta_2(\varepsilon). \tag{F.8}$$

Also, using our choice of $\gamma = \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/(8\|\mathbf{Z}\mathbf{Z}^\intercal\|_2)$, we have

$$\begin{aligned}
-4\gamma(\|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon) &= -4\cdot\frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{8\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}(\|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon) \\
&= -\frac{1}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\left(1 + \frac{\varepsilon}{\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}\right).
\end{aligned}$$

Considering a probability bound and using the result on $\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2$ in (F.5), we have

$$\begin{aligned}
&\mathbb{P}\left\{-2\gamma\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2 \leq -\frac{1}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\left(1 + \frac{\varepsilon}{\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}\right)\right\} \\
&= \mathbb{P}\big\{-2\gamma\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2 \leq -4\gamma(\|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon)\big\} \\
&= \mathbb{P}\big\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\big\|_2 \geq 2(\|\mathbf{Z}\mathbf{Z}^\intercal\|_2 + \varepsilon)\big\} \\
&\leq \delta_1(\varepsilon).
\end{aligned} \tag{F.9}$$

Combining (F.8) and (F.9) and using the union bound we obtain that

$$\begin{aligned}
-2\gamma\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal\|_2 + \min_{k\in[M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k &\geq -\frac{1}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\left(1 + \frac{\varepsilon}{\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}\right) + \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \varepsilon \\
&= \frac{1}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \left(1 + \frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{2\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}\right)\varepsilon
\end{aligned}$$

takes place with high probability at least $1 - [\delta_1(\varepsilon) + \delta_2(\varepsilon)]$. Recalling (F.7) we have

$$\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) = \min_{w\in\mathcal{S}_{n+m-1}} w^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w \geq -2\gamma\|\hat{\mathbf{Z}}\|_2^2 + \min_{k\in[M_\gamma]} w_k^\intercal \hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal w_k,$$

and thus the probability bound

$$\mathbb{P}\left\{\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) < \frac{1}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) - \left(1 + \frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{2\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}\right)\varepsilon\right\} \leq \delta_1(\varepsilon) + \delta_2(\varepsilon) =: \delta_m(\varepsilon). \tag{F.10}$$

In fact, this means that $\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal$ is invertible with probability at least $1-\delta_m(\varepsilon)$, if we take ε small enough, since $\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) > 0$ from Assumption 1 (iv). We conclude the proof by returning to the overall bound in (F.1) and using both the bound of $\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal$ in (F.2) and the bound of $\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)$ in (F.10) to obtain

$$\begin{aligned}
&\mathbb{P}\big\{\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\left\{\left[\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \geq \varepsilon\right] \bigcap \left[\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) \geq \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right]\right\} + \mathbb{P}\left\{\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) < \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right\} \\
&= \mathbb{P}\left\{\left[\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^{-1} - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \geq \varepsilon\right] \bigcap \left[\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) \geq \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right]\right\} + \mathbb{P}\left\{\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) < \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right\} \\
&\leq \mathbb{P}\left\{\left[\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \varepsilon\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)\right] \bigcap \left[\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) \geq \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right]\right\} \\
&\quad + \mathbb{P}\left\{\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) < \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right\} \\
&\leq \mathbb{P}\left\{\big\|\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal - \mathbf{Z}\mathbf{Z}^\intercal\big\|_2 \geq \frac{1-\tau}{2}\varepsilon\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)\right\} + \mathbb{P}\left\{\lambda_{\min}(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal) < \frac{1-\tau}{2}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\right\} \\
&\leq \delta_0\left(\frac{1-\tau}{2}\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)\varepsilon\right) + \delta_m\left(\frac{\tau\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{2+\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\|\mathbf{Z}\mathbf{Z}^\intercal\|_2}\right),
\end{aligned}$$

where $0 < \tau < 1$. Note that τ may be chosen arbitrarily small. In order to preserve useful dependence of the bound on ε, fix a maximum $\varepsilon_{\max} > \varepsilon$ and set $\tau = \varepsilon/\varepsilon_{\max}$, so the bound becomes

$$\begin{aligned}
&\mathbb{P}\big\{\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \geq \varepsilon\big\} \\
&\leq \delta_0\left(\frac{1}{2}\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)\left(1-\frac{\varepsilon}{\varepsilon_{\max}}\right)\varepsilon\right) + \delta_m\left(\frac{\varepsilon\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\varepsilon_{\max}(2+\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\|\mathbf{Z}\mathbf{Z}^\intercal\|_2)}\right) =: \delta_{ZZ}(\varepsilon,\varepsilon_{\max}).
\end{aligned} \tag{F.11}$$

Theorem 4 (Theorem 2 restated) *Suppose Assumptions 1 and 2 hold. Given a positive value $\varepsilon_{\max}$, then for all* $0 < \varepsilon < 3\varepsilon_{\max}\min\{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}, \varepsilon_{\max}\}$,

$$\mathbb{P}\big\{\big\|[\hat{A}\ \hat{B}] - [A\ B]\big\|_2 \geq \varepsilon\big\} \leq \delta_{AB}(\varepsilon),$$

where

$$\begin{aligned}
&\delta_{AB}(\varepsilon) = \delta_{AB}(\varepsilon,\varepsilon_{\max}) \\
&\quad := \delta_{YZ}\left(\frac{1}{3}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\varepsilon\right) + \delta_{YZ}\left(\sqrt{\frac{\varepsilon}{3}}\right) + \delta_{ZZ}\left(\frac{\varepsilon}{3\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}, \varepsilon_{\max}\right) + \delta_{ZZ}\left(\sqrt{\frac{\varepsilon}{3}}, \varepsilon_{\max}\right).
\end{aligned}$$

PROOF. Decompose the system parameter error matrix using the least-squares estimators, as discussed earlier, as

$$\begin{aligned}
[\hat{A}\ \hat{B}] - [A\ B] &= \hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - \mathbf{Y}\mathbf{Z}^\intercal(\mathbf{Z}\mathbf{Z}^\intercal)^{-1} \\
&= \underbrace{\big[\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big](\mathbf{Z}\mathbf{Z}^\intercal)^{-1}}_{=:\Pi_1} + \underbrace{\mathbf{Y}\mathbf{Z}^\intercal\big[(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big]}_{=:\Pi_2} + \underbrace{\big[\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big]\big[(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big]}_{=:\Pi_3}.
\end{aligned}$$

Considering a probability bound and using (E.4) we obtain

$$\begin{aligned}
\mathbb{P}\left\{[\hat{A}\ \hat{B}] - [A\ B] \geq \varepsilon\right\} &= \mathbb{P}\left\{\Pi_1 + \Pi_2 + \Pi_3 \geq \varepsilon\right\} \\
&\leq \mathbb{P}\left\{\Pi_1 \geq \varepsilon/3\right\} + \mathbb{P}\left\{\Pi_2 \geq \varepsilon/3\right\} + \mathbb{P}\left\{\Pi_3 \geq \varepsilon/3\right\}.
\end{aligned}$$

For the first term, use the submultiplicative property, rearrange, and use (E.1) to obtain

$$\mathbb{P}\left\{\Pi_1 \geq \frac{\varepsilon}{3}\right\} = \mathbb{P}\left\{\big\|\big[\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big](\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \geq \frac{\varepsilon}{3}\right\}$$

$$
\begin{aligned}
&\le \mathbb{P}\left\{\big\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big\|_2 \big\|(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \ge \frac{\varepsilon}{3}\right\} \\
&= \mathbb{P}\left\{\big\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big\|_2 \ge \frac{\varepsilon}{3\|(\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\|_2}\right\} \\
&= \mathbb{P}\left\{\big\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big\|_2 \ge \frac{1}{3}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\varepsilon\right\} \\
&\le \delta_{YZ}\left(\frac{1}{3}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\varepsilon\right),
\end{aligned}
$$

where the last step follows by applying Lemma 6 with the appropriate setting of ε.

For the second term, use submultiplicativity and rearrange to obtain

$$
\begin{aligned}
\mathbb{P}\left\{\Pi_2 \ge \frac{\varepsilon}{3}\right\} &= \mathbb{P}\left\{\big\|\mathbf{Y}\mathbf{Z}^\intercal\big[(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big]\big\|_2 \ge \frac{\varepsilon}{3}\right\} \\
&\le \mathbb{P}\left\{\|\mathbf{Y}\|_2\|\mathbf{Z}\|_2\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \ge \frac{\varepsilon}{3}\right\} \\
&= \mathbb{P}\left\{\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \ge \frac{\varepsilon}{3\|\mathbf{Y}\|_2\|\mathbf{Z}\|_2}\right\} \\
&\le \delta_{ZZ}\left(\frac{\varepsilon}{3\|\mathbf{Y}\|_2\|\mathbf{Z}\|_2}, \varepsilon_{\max}\right),
\end{aligned}
$$

where the last step follows by applying Lemma 7 with the appropriate setting of ε.

For the third term, use submultiplicativity and (E.5) to obtain

$$
\begin{aligned}
\mathbb{P}\left\{\Pi_3 \ge \frac{\varepsilon}{3}\right\} &= \mathbb{P}\left\{\big\|\big[\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big]\big[(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big]\big\|_2 \ge \frac{\varepsilon}{3}\right\} \\
&\le \mathbb{P}\left\{\big\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big\|_2\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \ge \frac{\varepsilon}{3}\right\} \\
&\le \mathbb{P}\left\{\big\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\big\|_2 \ge \sqrt{\frac{\varepsilon}{3}}\right\} + \mathbb{P}\left\{\big\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\big\|_2 \ge \sqrt{\frac{\varepsilon}{3}}\right\} \\
&\le \delta_{YZ}\left(\sqrt{\frac{\varepsilon}{3}}\right) + \delta_{ZZ}\left(\sqrt{\frac{\varepsilon}{3}}, \varepsilon_{\max}\right),
\end{aligned}
$$

where the last step follows by applying Lemmas 6 and 7 with the appropriate settings of ε. The conclusion follows by combining the probability bounds for each term.

PROOF OF THEOREM 2.

The qualitative claim in Theorem 2 is found by inverting the bound of Theorem 4 and examining the behavior of the bound as $n_r \to \infty$. To be specific, from Lemma 5, given fixed $\delta \in (0,1)$, we can find $\varepsilon_Y(\delta)$ such that

$$
\mathbb{P}\big\{\big\|\hat{\mathbf{Y}} - \mathbf{Y}\big\|_2 \ge \varepsilon_Y(\delta)\big\} \le \delta,
$$

where $\varepsilon_Y(\delta)$ satisfies

$$
\delta_Y(\varepsilon_Y(\delta)) = (n+\ell)\exp\left\{-\frac{3}{2}\cdot\frac{n_r\varepsilon_Y^2(\delta)}{3\ell c_N^2 + \varepsilon_Y(\delta)\sqrt{\ell c_N^2}}\right\} = \delta.
$$

Solving for $\varepsilon_Y(\delta)$ in terms of δ using the quadratic formula, we obtain

$$
\varepsilon_Y(\delta) = \frac{1}{2n_r}\left(\frac{2}{3}\sqrt{\ell c_N^2}\log\frac{n+\ell}{\delta} \pm \sqrt{\frac{4}{9}\ell c_N^2\log^2\frac{n+\ell}{\delta} + 8n_r\ell c_N^2\log\frac{n+\ell}{\delta}}\right).
$$

Since $\varepsilon_Y(\delta) \geq 0$, we have that

$$\begin{aligned}
\varepsilon_Y(\delta) &= \frac{1}{2n_r}\left(\frac{2}{3}\sqrt{\ell c_N^2}\log\frac{n+\ell}{\delta} + \sqrt{\frac{4}{9}\ell c_N^2\log^2\frac{n+\ell}{\delta} + 8n_r\ell c_N^2\log\frac{n+\ell}{\delta}}\right) \\
&= \frac{1}{3n_r}\sqrt{\ell c_N^2}\log\frac{n+\ell}{\delta} + \sqrt{\frac{1}{9n_r^2}\ell c_N^2\log^2\frac{n+\ell}{\delta} + \frac{2}{n_r}\ell c_N^2\log\frac{n+\ell}{\delta}} \\
&= \mathcal{O}\left(\sqrt{\frac{\ell c_N^2\log[(n+\ell)/\delta]}{n_r}}\right).
\end{aligned}$$

An identical argument holds for $\|\hat{\mathbf{Z}} - \mathbf{Z}\|_2$. Therefore, for fixed $\delta \in (0,1)$, with probability at least $1-\delta$,

$$\begin{aligned}
\|\hat{\mathbf{Y}} - \mathbf{Y}\|_2 < \varepsilon_Y(\delta) = \mathcal{O}\left(\sqrt{\frac{\ell c_N^2\log[(n+\ell)/\delta]}{n_r}}\right), \\
\|\hat{\mathbf{Z}} - \mathbf{Z}\|_2 < \varepsilon_Y(\delta) = \mathcal{O}\left(\sqrt{\frac{\ell c_N^2\log[(n+\ell)/\delta]}{n_r}}\right).
\end{aligned}$$

We proceed with this argument. Now from Lemma 6 it follows that for fixed $\delta \in (0,1)$ there is $\varepsilon_{YZ}(\delta)$ such that

$$\mathbb{P}\{\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\|_2 \geq \varepsilon_{YZ}(\delta)\} \leq \delta_{YZ}(\varepsilon_{YZ}(\delta)) = \delta.$$

This implies

$$\delta_{YZ}(\varepsilon_{YZ}(\delta)) = \delta_Y\left(\sqrt{\varepsilon_{YZ}(\delta) + \left(\frac{\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2}{2}\right)^2} - \frac{\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2}{2}\right) = \delta,$$

and solving for $\varepsilon_{YZ}(\delta)$ we obtain

$$\varepsilon_{YZ}(\delta) = \varepsilon_Y^2(\delta) + (\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2)\varepsilon_Y(\delta).$$

Thus when n_r is large enough it holds that

$$\varepsilon_{YZ}(\delta) = \mathcal{O}\big((\|\mathbf{Y}\|_2 + \|\mathbf{Z}\|_2)\varepsilon_Y(\delta)\big) = \mathcal{O}\left(\left(\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)} + \sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\sqrt{\frac{\ell c_N^2\log[(n+\ell)/\delta]}{n_r}}\right),$$

which is the bound of $\|\hat{\mathbf{Y}}\hat{\mathbf{Z}}^\intercal - \mathbf{Y}\mathbf{Z}^\intercal\|_2$ with probability at least $1-\delta$.

Similarly, under the condition of Lemma 7, we have that for fixed $\delta \in (0,1)$ and large enough n_r, there are $\varepsilon_0(\delta), \varepsilon_m(\delta) > 0$ with $\delta_0(\varepsilon_0(\delta)) = \delta$ and $\delta_m(\varepsilon_m(\delta)) = \delta$ such that

$$\begin{aligned}
\varepsilon_0(\delta) &= \varepsilon_Y^2(\delta) + 2\varepsilon_Y(\delta)\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)} = \mathcal{O}\left(\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2\log[(n+\ell)/\delta]}{n_r}}\right), \\
\varepsilon_m(\delta) &= \mathcal{O}\left(\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2\log\{(n+\ell)[9^{n+m} + (16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)+1)^{n+m}]/\delta\}}{n_r}}\right)
\end{aligned}$$

For fixed $\delta, \varepsilon_{\max} \in (0,1)$ and $\varepsilon_{Z1}(\delta) \in (0, \varepsilon_{\max})$, suppose that

$$\delta_0 \left(\frac{1}{2} \lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal) \left(1 - \frac{\varepsilon_{Z1}(\delta)}{\varepsilon_{\max}}\right) \varepsilon_{Z1}(\delta) \right) = \delta,$$

and we know that

$$\frac{1}{2} \lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal) \left(1 - \frac{\varepsilon_{Z1}(\delta)}{\varepsilon_{\max}}\right) \varepsilon_{Z1}(\delta) = \varepsilon_0(\delta),$$

which implies

$$\varepsilon_{Z1}(\delta) = \frac{1}{2} \varepsilon_{\max} \left(1 \pm \sqrt{1 - \frac{8\varepsilon_0(\delta)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}} \right).$$

Because for any nonnegative random variable X and constants $a > b > 0$, $\mathbb{P}\{X \geq a\} \leq \mathbb{P}\{X \geq b\}$, we choose the smaller root. Thus,

$$\begin{aligned}
\varepsilon_{Z1}(\delta) &= \frac{1}{2} \varepsilon_{\max} \left(1 - \sqrt{1 - \frac{8\varepsilon_0(\delta)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}} \right) \\
&= \frac{1}{2} \varepsilon_{\max} \left(1 - \left(1 - \mathcal{O}\left(\frac{\varepsilon_0(\delta)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)} \right) \right) \right) \\
&= \mathcal{O}\left(\frac{\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)} \sqrt{\frac{\ell c_N^2 \log[(n+\ell)/\delta]}{n_r}} \right).
\end{aligned}$$

For $\delta, \varepsilon_{\max} \in (0,1)$ and $\varepsilon_{Z2}(\delta) \in (0, \varepsilon_{\max})$, if

$$\delta_m \left(\frac{\varepsilon_{Z2}(\delta) \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal))} \right) = \delta,$$

then

$$\begin{aligned}
\varepsilon_{Z2}(\delta) &= \frac{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal))}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)} \varepsilon_m(\delta) \\
&= \mathcal{O}\left(\left(2 + \frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)} \right) \frac{\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)} \sqrt{\frac{\ell c_N^2 \log\{(n+\ell)[9^{n+m} + (16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal) + 1)^{n+m}]/\delta\}}{n_r}} \right)
\end{aligned}$$

Note that $c_1\varepsilon^2/(c_2 + c_3\varepsilon)$ is monotonically increasing on $(0, +\infty)$ for any positive constants c_1, c_2, and c_3, so from the monotonicity of composite functions, we know that $\delta_Y(\varepsilon)$ is monotonically decreasing. So are $\delta_{YZ}(\varepsilon)$, $\delta_0(\varepsilon)$, and $\delta_m(\varepsilon)$. In addition, $(1 - \varepsilon/\varepsilon_{\max})\varepsilon$ is monotonically increasing on $(0, \varepsilon_{\max}/2)$ for fixed $\varepsilon_{\max} > 0$, implying $\delta_{ZZ}(\varepsilon, \varepsilon_{\max})$ is monotonically decreasing on $(0, \varepsilon_{\max}/2)$. Let $\varepsilon_{ZZ}(\delta) := \max\{\varepsilon_{Z1}(\delta/2), \varepsilon_{Z2}(\delta/2)\}$. Since for fixed $\delta, \varepsilon_{\max} \in (0,1)$, when n_r is large enough, $\varepsilon_{Z1}(\delta/2), \varepsilon_{Z2}(\delta/2) < \varepsilon_{\max}/2$, it holds that

$$\begin{aligned}
&\mathbb{P}\{\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\|_2 \geq \varepsilon_{ZZ}(\delta)\} \\
&\leq \delta_{ZZ}(\varepsilon_{ZZ}(\delta), \varepsilon_{\max}) \\
&= \delta_0 \left(\frac{1}{2} \lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal) \left(1 - \frac{\varepsilon_{ZZ}(\delta)}{\varepsilon_{\max}}\right) \varepsilon_{ZZ}(\delta) \right) + \delta_m \left(\frac{\varepsilon_{ZZ}(\delta) \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal))} \right) \\
&\leq \delta_0 \left(\frac{1}{2} \lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal) \left(1 - \frac{\varepsilon_{Z1}(\delta/2)}{\varepsilon_{\max}}\right) \varepsilon_{Z1}(\delta/2) \right) + \delta_m \left(\frac{\varepsilon_{Z2}(\delta/2) \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal))} \right) = \delta.
\end{aligned}$$

Therefore, we know that with probability at least $1-\delta$,

$$
\begin{aligned}
&\left\|(\hat{\mathbf{Z}}\hat{\mathbf{Z}}^\intercal)^\dagger - (\mathbf{Z}\mathbf{Z}^\intercal)^{-1}\right\|_2 \\
&< \varepsilon_{ZZ}(\delta) \\
&= \mathcal{O}\Bigg(\max\Bigg\{ \frac{\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log[2(n+\ell)/\delta]}{n_r}},
\end{aligned}
$$

$$
\left(2+\frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\frac{\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log\{2(n+\ell)[9^{n+m}+(16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)+1)^{n+m}]/\delta\}}{n_r}}\Bigg\}\Bigg).
$$

Similarly, let

$$
\varepsilon_{AB}(\delta) := \max\left\{\frac{3\varepsilon_{YZ}(\delta/4)}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}, 3\varepsilon_{YZ}^2(\delta/4), 3\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\varepsilon_{ZZ}(\delta/4), 3\varepsilon_{ZZ}^2(\delta/4)\right\}, \tag{F.12}
$$

and from Theorem 4 it holds for large enough n_r that

$$
\begin{aligned}
&\mathbb{P}\left\{\left\|[\hat{A}\ \hat{B}] - [A\ B]\right\|_2 \geq \varepsilon_{AB}(\delta)\right\} \\
&\leq \delta_{AB}(\varepsilon_{AB}(\delta)) \\
&= \delta_{YZ}\left(\frac{1}{3}\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)\varepsilon_{AB}(\delta)\right) + \delta_{YZ}\left(\sqrt{\frac{\varepsilon_{AB}(\delta)}{3}}\right) + \delta_{ZZ}\left(\frac{\varepsilon_{AB}(\delta)}{3\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}, \varepsilon_{\max}\right) \\
&\quad + \delta_{ZZ}\left(\sqrt{\frac{\varepsilon_{AB}(\delta)}{3}}, \varepsilon_{\max}\right) \\
&\leq \delta_{YZ}(\varepsilon_{YZ}(\delta/4)) + \delta_{YZ}(\varepsilon_{YZ}(\delta/4)) + \delta_{ZZ}(\varepsilon_{ZZ}(\delta/4), \varepsilon_{\max}) + \delta_{ZZ}(\varepsilon_{ZZ}(\delta/4), \varepsilon_{\max}) = \delta.
\end{aligned}
$$

Therefore, we have that with probability $1-\delta$, for large enough n_r,

$$
\left\|[\hat{A}\ \hat{B}] - [A\ B]\right\|_2 < \varepsilon_{AB}(\delta),
$$

where

$$
\begin{aligned}
&\varepsilon_{AB}(\delta) \\
&= \mathcal{O}\Bigg(\max\Bigg\{ \frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}+\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log[4(n+\ell)/\delta]}{n_r}},
\end{aligned}
$$

$$
\left(\left(\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}+\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\sqrt{\frac{\ell c_N^2 \log[4(n+\ell)/\delta]}{n_r}}\right)^2, \frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log[8(n+\ell)/\delta]}{n_r}},
$$

$$
\left(2+\frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log\{8(n+\ell)[9^{n+m}+(16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)+1)^{n+m}]/\delta\}}{n_r}},
$$

$$
\left(\frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log[8(n+\ell)/\delta]}{n_r}}\right)^2,
$$

$$
\Bigg(\left(2+\frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}
$$

$$
\sqrt{\frac{\ell c_N^2 \log\{8(n+\ell)[9^{n+m}+(16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)+1)^{n+m}]/\delta\}}{n_r}}\Bigg)^2\Bigg\}\Bigg)
$$

$$
= \mathcal{O}\Bigg(\max\Bigg\{ \frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}+\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log[4(n+\ell)/\delta]}{n_r}}, \frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2 \log[8(n+\ell)/\delta]}{n_r}},
$$

$$
\left(2+\frac{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2\log\{8(n+\ell)[9^{n+m}+(16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)+1)^{n+m}]/\delta\}}{n_r}}\Bigg\}\Bigg)
$$

$$
=\mathcal{O}\Bigg(\max\Bigg\{\frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}+\sqrt{\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2\log[4(n+\ell)/\delta]}{n_r}},\frac{\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}^2(\mathbf{Z}\mathbf{Z}^\intercal)}\sqrt{\frac{\ell c_N^2\log[8(n+\ell)/\delta]}{n_r}},
$$

$$
\sqrt{\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)}\left(1+\frac{2\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)}{\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)}\right)\sqrt{\frac{\ell c_N^2\log\{8(n+\ell)[9^{n+m}+(16\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)/\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)+1)^{n+m}]/\delta\}}{n_r}}\Bigg\}\Bigg).
$$

The qualitative claim in Theorem 2 follows by dropping dependence on quantities other than δ, n_r, and ℓ.

Remark 14 *It can be seen from the above bound that smaller $\lambda_{\max}(\mathbf{Y}\mathbf{Y}^\intercal)$, $\lambda_{\max}(\mathbf{Z}\mathbf{Z}^\intercal)$, c_N, and larger $\lambda_{\min}(\mathbf{Z}\mathbf{Z}^\intercal)$, all of which depend on both system parameters and input design, yield faster convergence speed of Algorithm 1. The exponential term of $n+m$ is technical and could be tightened [45, 46]. In addition, the estimation error has higher order terms, e.g., $\mathcal{O}(1/n_r)$, which may be relatively large when n_r is small. This could explain the performance of Algorithm 1 with small number of rollouts in simulation.*

G Proof of Theorem 3

Now we derive bounds for $[\hat{\tilde{\Sigma}}'_A \ \hat{\tilde{\Sigma}}'_B] - [\tilde{\Sigma}'_A \ \tilde{\Sigma}'_B]$. The proofs follow a similar structure to that of the proofs for bounds on $[\hat{A} \ \hat{B}] - [A \ B]$, but with more complicated expressions due to the greater complexity of the second-moment dynamic.

Throughout this section, small probability bounds are denoted by $\eta_{[\cdot]}$, where $[\cdot]$ are various subscripts, and each of these bounds decreases monotonically towards 0 with increasing number of rollouts n_r.

Recalling notations in Section 3, we have

$$
\begin{aligned}
&\hat{\tilde{X}}_t = \frac{1}{n_r} P_1 \operatorname{vec}\left(\sum_{k=1}^{n_r} x_t^{(k)} (x_t^{(k)})^\intercal\right),\ \tilde{X}_t = P_1 \operatorname{vec}(\mathbb{E}\{x_t x_t^\intercal\}),\\
&\tilde{U}_t = P_2 \operatorname{vec}(\bar{U}_t + \nu_t \nu_t^\intercal),\\
&\hat{W}_t = \frac{1}{n_r} \operatorname{vec}\left(\sum_{k=1}^{n_r} x_t^{(k)} \nu_t^\intercal\right),\ W_t = \operatorname{vec}(\mathbb{E}\{x_t u_t^\intercal\}),\\
&\hat{W}'_t = \frac{1}{n_r} \operatorname{vec}\left(\sum_{k=1}^{n_r} \nu_t (x_t^{(k)})^\intercal\right),\ W'_t = \operatorname{vec}(\mathbb{E}\{u_t^\intercal x_t\}),\\
&\hat{\tilde{A}} = P_1(\hat{A} \otimes \hat{A}) Q_1, \quad \tilde{A} = P_1(A \otimes A) Q_1,\\
&\hat{\tilde{B}} = P_1(\hat{B} \otimes \hat{B}) Q_2, \quad \tilde{B} = P_1(B \otimes B) Q_2,\\
&\hat{K}_{BA} = P_1(\hat{B} \otimes \hat{A}), \quad K_{BA} = P_1(B \otimes A),\\
&\hat{K}_{AB} = P_1(\hat{A} \otimes \hat{B}), \quad K_{AB} = P_1(A \otimes B).
\end{aligned}
$$

Further denote

$$
\mathbf{M}_1 := \left[\tilde{X}_{\ell-1} \cdots \tilde{X}_0\right],\ \mathbf{L}_1 := \left[W_{\ell-1} \cdots W_0\right],\ \mathbf{U} := \left[\tilde{U}_{\ell-1} \cdots \tilde{U}_0\right].
$$

Lemma 8 *Suppose Assumptions 1 and 2 hold. Then for all $\varepsilon > 0$,*

$$
\mathbb{P}\left\{\left\|\hat{\mathbf{D}} - \mathbf{D}\right\|_2 \geq \varepsilon\right\} \leq \eta_D(\varepsilon),
$$

where

$$
\eta_D(\varepsilon) := \left(\frac{n(n+1)}{2} + \ell\right) \exp\left\{-\frac{3}{2} \cdot \frac{n_r \varepsilon^2}{3\ell c_F^2 + \varepsilon\sqrt{\ell c_F^2}}\right\},
$$

and

$$
\mathbb{P}\left\{\left\|\hat{\mathbf{C}} - \mathbf{C}\right\|_2 \geq \varepsilon\right\} \leq \eta_C(\varepsilon),
$$

where

$$
\begin{aligned}
\eta_C(\varepsilon) &:= \eta_D(\varepsilon/5) + \eta_{AM}(\varepsilon/5) + 2\eta_{KL}(\varepsilon/5) + \eta_B\left(\frac{\varepsilon}{5\|\mathbf{U}\|_2}\right),\\
\eta_{AM}(\varepsilon) &:= \eta_A\left(\frac{\varepsilon}{3\|\mathbf{M}_1\|_2}\right) + \eta_D\left(\frac{\varepsilon}{6\|A\|_2^2}\right) + \eta_A(\sqrt{\varepsilon/3}) + \eta_D(\sqrt{\varepsilon/3}),\\
\eta_{KL}(\varepsilon) &:= \eta_{AB}\left(\frac{\varepsilon}{3\|\mathbf{L}_1\|_2}\right) + \eta_L\left(\frac{\varepsilon}{3\|A\|_2\|B\|_2}\right) + \eta_{AB}\left(\sqrt{\varepsilon/3}\right) + \eta_L\left(\sqrt{\varepsilon/3}\right),\\
\eta_A(\varepsilon) &:= \delta_{AB}\left(\sqrt{\varepsilon}/2\right) + \delta_{AB}\left(\varepsilon/(8\sqrt{\|A\|_2})\right),
\end{aligned}
$$

$$\begin{aligned}
\eta_B(\varepsilon) &:= \delta_{AB}\left(\sqrt{\varepsilon}/2\right) + \delta_{AB}\left(\varepsilon/(8\sqrt{\|B\|_2})\right), \\
\eta_{AB}(\varepsilon) &:= 2\delta_{AB}\left(\sqrt{\varepsilon/3}\right) + \delta_{AB}\left(\varepsilon/(3\sqrt{\|B\|_2})\right) + \delta_{AB}\left(\varepsilon/(3\sqrt{\|A\|_2})\right), \\
\eta_L(\varepsilon) &:= (nm+\ell)\exp\left\{-\frac{3}{2}\cdot\frac{n_r\varepsilon^2}{3\ell c_W^2 + \varepsilon\sqrt{\ell c_W^2}}\right\}.
\end{aligned}$$

Here, $\delta_{AB}(\varepsilon) = \delta_{AB}(\varepsilon, \varepsilon_{\max})$, $\varepsilon \in (0, \varepsilon_{\max})$ and $\varepsilon_{\max} \in (0,1)$, is defined in Theorem 4, and we omit $\varepsilon_{\max}$ for simplicity.

PROOF. Denote

$$\begin{aligned}
\mathbf{M}_1 &:= \left[\tilde{X}_{\ell-1}\cdots\tilde{X}_0\right], & \hat{\mathbf{M}}_1 &:= \left[\hat{\tilde{X}}_{\ell-1}\cdots\hat{\tilde{X}}_0\right], \\
\mathbf{M}_2 &:= \left[\tilde{X}_{\ell}\cdots\tilde{X}_1\right], & \hat{\mathbf{M}}_2 &:= \left[\hat{\tilde{X}}_{\ell}\cdots\hat{\tilde{X}}_1\right], \\
\mathbf{L}_1 &:= \left[W_{\ell-1}\cdots W_0\right], & \hat{\mathbf{L}}_1 &:= \left[\hat{W}_{\ell-1}\cdots\hat{W}_0\right], \\
\mathbf{L}_2 &:= \left[W'_{\ell-1}\cdots W'_0\right], & \hat{\mathbf{L}}_2 &:= \left[\hat{W}'_{\ell-1}\cdots\hat{W}'_0\right], \\
\mathbf{U} &:= \left[\tilde{U}_{\ell-1}\cdots\tilde{U}_0\right],
\end{aligned}$$

which will be used both for the development of the bound on $\|\hat{\mathbf{C}} - \mathbf{C}\|_2$ and on $\|\hat{\mathbf{D}} - \mathbf{D}\|_2$.

We begin by justifying the claim regarding a bound on $\|\hat{\mathbf{D}} - \mathbf{D}\|_2$. We make the new definitions

$$\begin{aligned}
\mathbf{X}_k &:= \left[\mathrm{vec}\left(x_{\ell-1}^{(k)}(x_{\ell-1}^{(k)})^\intercal - \mathbb{E}\{x_{\ell-1}^{(k)}(x_{\ell-1}^{(k)})^\intercal\}\right)\ \cdots\ \mathrm{vec}\left(x_0^{(k)}(x_0^{(k)})^\intercal - \mathbb{E}\{x_0^{(k)}(x_0^{(k)})^\intercal\}\right)\right], \\
\widetilde{\mathbf{X}}_k &:= P_1\mathbf{X}_k,
\end{aligned}$$

so that

$$\hat{\mathbf{M}}_1 - \mathbf{M}_1 = \frac{1}{n_r}\sum_{k=1}^{n_r}\widetilde{\mathbf{X}}_k.$$

Considering a single column of $\mathbf{X}_k$, we use the bound from Lemma 1 to obtain

$$\begin{aligned}
\|\widetilde{\mathbf{X}}_k\|_2 &\le \|P_1\|_2\|\mathbf{X}_k\|_2 && \text{(by submultiplicativity)} \\
&= \|\mathbf{X}_k\|_2 && \text{(since } \|P_1\|_2 = 1 \text{ by definition of } P_1\text{)} \\
&\le \|\mathbf{X}_k\|_F && \text{(by ordering of } \|\cdot\|_2 \text{ and } \|\cdot\|_F\text{)} \\
&= \sqrt{\sum_{t=0}^{\ell-1}\left\|\mathrm{vec}\left(x_t^{(k)}(x_t^{(k)})^\intercal - \mathbb{E}\{x_t^{(k)}(x_t^{(k)})^\intercal\}\right)\right\|^2} && \text{(by definition of } \mathbf{X}_k\text{, vec, } \|\cdot\|_F\text{)} \\
&\le \sqrt{\ell c_F^2}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\|\hat{\mathbf{D}} - \mathbf{D}\|_2 &= \left\|\begin{bmatrix}\hat{\tilde{X}}_{\ell-1} & \cdots & \hat{\tilde{X}}_0 \\ \tilde{U}_{\ell-1} & \cdots & \tilde{U}_0\end{bmatrix} - \begin{bmatrix}\tilde{X}_{\ell-1} & \cdots & \tilde{X}_0 \\ \tilde{U}_{\ell-1} & \cdots & \tilde{U}_0\end{bmatrix}\right\|_2 \\
&= \left\|\left[\hat{\tilde{X}}_{\ell-1}\ \cdots\ \hat{\tilde{X}}_0\right] - \left[\tilde{X}_{\ell-1}\ \cdots\ \tilde{X}_0\right]\right\|_2 \\
&= \left\|\frac{1}{n_r}\sum_{k=1}^{n_r}\widetilde{\mathbf{X}}_k\right\|_2 \qquad (G.1)
\end{aligned}$$

Thus we have the small probability bound

$$\mathbb{P}\{\|\hat{\mathbf{D}} - \mathbf{D}\|_2 \geq \varepsilon\} \leq \eta_D(\varepsilon),$$

where

$$\eta_D(\varepsilon) := \left(\frac{1}{2}n(n+1) + \ell\right) \exp\left\{-\frac{3}{2} \cdot \frac{n_r \varepsilon^2}{3\ell c_F^2 + \varepsilon\sqrt{\ell c_F^2}}\right\},$$

which follows by applying Corollary 2 with $Y_k = \widetilde{\mathbf{X}}_k$, $N = n_r$, and $M = \sqrt{\ell c_F^2}$.

We now justify the claim regarding a bound on $\|\hat{\mathbf{C}} - \mathbf{C}\|_2$. The prior statement implies

$$\mathbb{P}\{\|\hat{\mathbf{M}}_1 - \mathbf{M}_1\|_2 \geq \varepsilon\} \leq \eta_D(\varepsilon). \tag{G.2}$$

An identical argument, but shifting the time indices of all terms by 1, leads to the bound

$$\mathbb{P}\{\|\hat{\mathbf{M}}_2 - \mathbf{M}_2\|_2 \geq \varepsilon\} \leq \eta_D(\varepsilon). \tag{G.3}$$

We will also need probabilistic bounds on the cross-terms $\hat{\mathbf{L}}_1 - \mathbf{L}_1$ and $\hat{\mathbf{L}}_2 - \mathbf{L}_2$. To this end, make the new definition

$$\mathbf{W}_k := \left[\operatorname{vec}\left(x_{\ell-1}^{(k)}(u_{\ell-1}^{(k)})^\intercal - \mathbb{E}\{x_{\ell-1}^{(k)}(u_{\ell-1}^{(k)})^\intercal\}\right) \cdots \operatorname{vec}\left(x_0^{(k)}(u_0^{(k)})^\intercal - \mathbb{E}\{x_0^{(k)}(u_0^{(k)})^\intercal\}\right)\right],$$

so that

$$\hat{\mathbf{L}}_1 - \mathbf{L}_1 = \frac{1}{n_r}\sum_{k=1}^{n_r} \mathbf{W}_k.$$

Considering a single column of $\mathbf{W}_k$, we use the bound from Lemma 1 to obtain

$$\begin{aligned}
\|\mathbf{W}_k\|_2 &\leq \|\mathbf{W}_k\|_F && \text{(by ordering of } \|\cdot\|_2 \text{ and } \|\cdot\|_F) \\
&= \sqrt{\sum_{t=0}^{\ell-1} \left\|\operatorname{vec}\left(x_t^{(k)}(u_t^{(k)})^\intercal - \mathbb{E}\{x_t^{(k)}(u_t^{(k)})^\intercal\}\right)\right\|^2} && \text{(by definition of } \mathbf{W}_k\text{, vec, } \|\cdot\|_F) \\
&\leq \sqrt{\ell c_W^2}.
\end{aligned}$$

Thus we have the probability bound

$$\mathbb{P}\{\|\hat{\mathbf{L}}_1 - \mathbf{L}_1\|_2 \geq \varepsilon\} \leq \eta_L(\varepsilon), \tag{G.4}$$

where

$$\eta_L(\varepsilon) := (nm + \ell) \exp\left\{-\frac{3}{2} \cdot \frac{n_r \varepsilon^2}{3\ell c_W^2 + \varepsilon\sqrt{\ell c_W^2}}\right\},$$

which follows by applying Corollary 2 with $Y_k = \mathbf{W}_k$, $N = n_r$, and $M = \sqrt{\ell c_W^2}$. An identical argument yields the same bound for $\hat{\mathbf{L}}_2 - \mathbf{L}_2$, i.e.

$$\mathbb{P}\{\|\hat{\mathbf{L}}_2 - \mathbf{L}_2\|_2 \geq \varepsilon\} \leq \eta_L(\varepsilon). \tag{G.5}$$

Denote the optimal estimation error bounds on A and B as

$$\delta_{A,*}(\varepsilon) := \mathbb{P}\{\|\hat{A} - A\|_2 \geq \varepsilon\},\ \delta_{B,*}(\varepsilon) := \mathbb{P}\{\|\hat{B} - B\|_2 \geq \varepsilon\}.$$

By Theorem 2 we know $\delta_{A,*}(\varepsilon) \leq \delta_{AB}(\varepsilon)$ and $\delta_{B,*}(\varepsilon) \leq \delta_{AB}(\varepsilon)$, so we can use the computable bound $\delta_{AB}(\varepsilon)$ in Theorem 2 as a conservative approximation of $\delta_{A,*}(\varepsilon)$ and $\delta_{B,*}(\varepsilon)$.

From the assumption of Theorem 3, it holds that

$$\begin{aligned}
&\mathbb{P}\{\|\hat{\tilde{A}} - \tilde{A}\|_2 \geq \varepsilon\} \\
&= \mathbb{P}\{\|P_1(\hat{A} \otimes \hat{A})Q_1 - P_1(A \otimes A)Q_1\|_2 \geq \varepsilon\} && \text{(definition of } \hat{\tilde{A}},\ \tilde{A}) \\
&\leq \mathbb{P}\{\|P_1\|_2\|\hat{A} \otimes \hat{A} - A \otimes A\|_2\|Q_1\|_2 \geq \varepsilon\} && \text{(submultiplicativity)} \\
&\leq \mathbb{P}\{2\|\hat{A} \otimes \hat{A} - A \otimes A\|_2 \geq \varepsilon\} && \text{(by definition, } \|P_1\|_2 = 1,\ \|Q_1\|_2 \leq 2) \\
&= \mathbb{P}\{\|(\hat{A} - A) \otimes (\hat{A} - A) + (\hat{A} - A) \otimes A + A \otimes (\hat{A} - A)\|_2 \geq \varepsilon/2\} \\
&\leq \mathbb{P}\{\|(\hat{A} - A) \otimes (\hat{A} - A)\|_2 \geq \varepsilon/4\} + \mathbb{P}\{\|(\hat{A} - A) \otimes A + A \otimes (\hat{A} - A)\|_2 \geq \varepsilon/4\} && \text{(by E.4)} \\
&= \mathbb{P}\{\|\hat{A} - A\|_2 \geq \sqrt{\varepsilon}/2\} + \mathbb{P}\{\|\hat{A} - A\|_2 \geq \varepsilon/(8\|A\|_2)\} && (\|A \otimes B\|_2 = \|A\|_2\|B\|_2) \\
&= \delta_{A,*}\left(\sqrt{\varepsilon}/2\right) + \delta_{A,*}\left(\varepsilon/(8\sqrt{\|A\|})\right) \\
&\leq \delta_{AB}\left(\sqrt{\varepsilon}/2\right) + \delta_{AB}\left(\varepsilon/(8\sqrt{\|A\|})\right) && \text{(G.6)} \\
&=: \eta_A(\varepsilon), && \text{(G.7)}
\end{aligned}$$

and by an identical argument

$$\begin{aligned}
&\mathbb{P}\{\|\hat{\tilde{B}} - \tilde{B}\|_2 \geq \varepsilon\} \\
&\leq \delta_{B,*}\left(\sqrt{\varepsilon}/2\right) + \delta_{B,*}\left(\varepsilon/(8\sqrt{\|A\|})\right) \\
&\leq \delta_{AB}\left(\sqrt{\varepsilon}/2\right) + \delta_{AB}\left(\varepsilon/(8\sqrt{\|B\|})\right) =: \eta_B(\varepsilon). && \text{(G.8)}
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}\{\|\hat{K}_{AB} - K_{AB}\|_2 \geq \varepsilon\} \\
&= \mathbb{P}\{\|\hat{K}_{BA} - K_{BA}\|_2 \geq \varepsilon\} && \text{(by symmetry)} \\
&= \mathbb{P}\{\|P_1(\hat{A} \otimes \hat{B}) - P_1(A \otimes B)\|_2 \geq \varepsilon\} \\
&= \mathbb{P}\{\|P_1\|_2\|(\hat{A} \otimes \hat{B}) - (A \otimes B)\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|(\hat{A} - A) \otimes (\hat{B} - B) + (\hat{A} - A) \otimes B + A \otimes (\hat{B} - B)\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|(\hat{A} - A) \otimes (\hat{B} - B)\|_2 \geq \varepsilon/3\} + \mathbb{P}\{\|(\hat{A} - A) \otimes B\|_2 \geq \varepsilon/3\} + \mathbb{P}\{\|A \otimes (\hat{B} - B)\|_2 \geq \varepsilon/3\} \\
&\leq \mathbb{P}\{\|\hat{A} - A\|_2 \geq \sqrt{\varepsilon/3}\} + \mathbb{P}\{\|\hat{B} - B\|_2 \geq \sqrt{\varepsilon/3}\} + \mathbb{P}\{\|\hat{A} - A\|_2 \geq \varepsilon/(3\|B\|_2)\} + \mathbb{P}\{\|\hat{B} - B\|_2 \geq \varepsilon/(3\|A\|_2)\} \\
&= \delta_{A,*}\left(\sqrt{\varepsilon/3}\right) + \delta_{B,*}\left(\sqrt{\varepsilon/3}\right) + \delta_{A,*}\left(\varepsilon/(3\sqrt{\|B\|})\right) + \delta_{B,*}\left(\varepsilon/(3\sqrt{\|A\|})\right) \\
&\leq 2\delta_{AB}\left(\sqrt{\varepsilon/3}\right) + \delta_{AB}\left(\varepsilon/(3\sqrt{\|B\|})\right) + \delta_{AB}\left(\varepsilon/(3\sqrt{\|A\|})\right) && \text{(G.9)} \\
&=: \eta_{AB}(\varepsilon). && \text{(G.10)}
\end{aligned}$$

Consider the decomposition of $\hat{\mathbf{C}} - \mathbf{C}$ as

$$\begin{aligned}
&\hat{\mathbf{C}} - \mathbf{C} \\
&= \left([\hat{\tilde{X}}_\ell \cdots \hat{\tilde{X}}_1] - [\tilde{X}_\ell \cdots \tilde{X}_1]\right) - \left(\hat{\tilde{A}}[\hat{\tilde{X}}_{\ell-1} \cdots \hat{\tilde{X}}_0] - \tilde{A}[\tilde{X}_{\ell-1} \cdots \tilde{X}_0]\right)
\end{aligned}$$

$$
\begin{aligned}
&- \big(\hat{K}_{BA}\big[\hat{W}_{\ell-1}\cdots\hat{W}_0\big] - K_{BA}\big[W_{\ell-1}\cdots W_0\big]\big) - \big(\hat{K}_{AB}\big[\hat{W}'_{\ell-1}\cdots\hat{W}'_0\big] - K_{AB}\big[W'_{\ell-1}\cdots W'_0\big]\big) \\
&- \big(\hat{\tilde{B}}\big[\tilde{U}_{\ell-1}\cdots\tilde{U}_0\big] - \tilde{B}\big[\tilde{U}_{\ell-1}\cdots\tilde{U}_0\big]\big) \\
= &\big(\hat{\mathbf{M}}_2 - \mathbf{M}_2\big) - \big(\hat{\tilde{A}}\hat{\mathbf{M}}_1 - \tilde{A}\mathbf{M}_1\big) - \big(\hat{K}_{BA}\hat{\mathbf{L}}_1 - K_{BA}\mathbf{L}_1\big) - \big(\hat{K}_{AB}\hat{\mathbf{L}}_2 - K_{AB}\mathbf{L}_2\big) - \big(\hat{\tilde{B}} - \tilde{B}\big)\mathbf{U}. \qquad \text{(G.11)}
\end{aligned}
$$

We treat each of these five terms separately.

For the first term, $\hat{\mathbf{M}}_2 - \mathbf{M}_2$, we have the bound in (G.3).

For the second term, $\hat{\tilde{A}}\hat{\mathbf{M}}_1 - \tilde{A}\mathbf{M}_1$, we have the decomposition

$$
\hat{\tilde{A}}\hat{\mathbf{M}}_1 - \tilde{A}\mathbf{M}_1 = \big(\hat{\tilde{A}} - \tilde{A}\big)\mathbf{M}_1 + \tilde{A}\big(\hat{\mathbf{M}}_1 - \mathbf{M}_1\big) + \big(\hat{\tilde{A}} - \tilde{A}\big)\big(\hat{\mathbf{M}}_1 - \mathbf{M}_1\big).
$$

Considering a probability bound for each of these three subterms, we have

$$
\begin{aligned}
&\mathbb{P}\big\{\big\|(\hat{\tilde{A}} - \tilde{A})\mathbf{M}_1\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|\hat{\tilde{A}} - \tilde{A}\big\|_2 \|\mathbf{M}_1\|_2 \geq \varepsilon\big\} && \text{(by submultiplicativity)} \\
&= \mathbb{P}\left\{\big\|(\hat{\tilde{A}} - \tilde{A})\big\|_2 \geq \frac{\varepsilon}{\|\mathbf{M}_1\|_2}\right\} \\
&\leq \eta_A\left(\frac{\varepsilon}{\|\mathbf{M}_1\|_2}\right), && \text{(by (G.7))} \\
&\mathbb{P}\big\{\big\|\tilde{A}(\hat{\mathbf{M}}_1 - \mathbf{M}_1)\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|\tilde{A}\big\|_2 \big\|\hat{\mathbf{M}}_1 - \mathbf{M}_1\big\|_2 \geq \varepsilon\big\} && \text{(by submultiplicativity)} \\
&\leq \mathbb{P}\big\{2\|A\|_2^2 \big\|\hat{\mathbf{M}}_1 - \mathbf{M}_1\big\|_2 \geq \varepsilon\big\} && \text{(since } \big\|\tilde{A}\big\|_2 = \|P_1(A \otimes A)Q_1\|_2 \leq 2\|A \otimes A\|_2 = 2\|A\|_2^2\text{)} \\
&= \mathbb{P}\left\{\big\|\hat{\mathbf{M}}_1 - \mathbf{M}_1\big\|_2 \geq \frac{\varepsilon}{2\|A\|_2^2}\right\} \\
&\leq \eta_D\left(\frac{\varepsilon}{2\|A\|_2^2}\right), && \text{(by (G.2))}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\big\{\big\|(\hat{\tilde{A}} - \tilde{A})(\hat{\mathbf{M}}_1 - \mathbf{M}_1)\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|\hat{\tilde{A}} - \tilde{A}\big\|_2 \big\|\hat{\mathbf{M}}_1 - \mathbf{M}_1\big\|_2 \geq \varepsilon\big\} && \text{(by submultiplicativity)} \\
&\leq \mathbb{P}\big\{\big\|\hat{\tilde{A}} - \tilde{A}\big\|_2 \geq \sqrt{\varepsilon}\big\} + \mathbb{P}\big\{\big\|\hat{\mathbf{M}}_1 - \mathbf{M}_1\big\|_2 \geq \sqrt{\varepsilon}\big\} && \text{(by (E.5))} \\
&\leq \eta_A(\sqrt{\varepsilon}) + \eta_D(\sqrt{\varepsilon}). && \text{(by (G.7) and (G.2))}
\end{aligned}
$$

Putting together the bounds for the three subterms,

$$
\begin{aligned}
&\mathbb{P}\big\{\big\|\hat{\tilde{A}}\hat{\mathbf{M}}_1 - \tilde{A}\mathbf{M}_1\big\|_2 \geq \varepsilon\big\} \\
&\leq \mathbb{P}\big\{\big\|(\hat{\tilde{A}} - \tilde{A})\mathbf{M}_1\big\|_2 \geq \varepsilon/3\big\} + \mathbb{P}\big\{\big\|\tilde{A}(\hat{\mathbf{M}}_1 - \mathbf{M}_1)\big\|_2 \geq \varepsilon/3\big\} + \mathbb{P}\big\{\big\|(\hat{\tilde{A}} - \tilde{A})(\hat{\mathbf{M}}_1 - \mathbf{M}_1)\big\|_2 \geq \varepsilon/3\big\} && \text{(by (E.4))} \\
&\leq \eta_A\left(\frac{\varepsilon}{3\,\|\mathbf{M}_1\|_2}\right) + \eta_D\left(\frac{\varepsilon}{6\,\|A\|_2^2}\right) + \eta_A(\sqrt{\varepsilon/3}) + \eta_D(\sqrt{\varepsilon/3}) \\
&=: \eta_{AM}(\varepsilon).
\end{aligned}
$$

For the third term, $\hat{K}_{BA}\hat{\mathbf{L}}_1 - K_{BA}\mathbf{L}_1$, we have the decomposition

$$
\hat{K}_{BA}\hat{\mathbf{L}}_1 - K_{BA}\mathbf{L}_1 = (\hat{K}_{BA} - K_{BA})\mathbf{L}_1 + K_{BA}(\hat{\mathbf{L}}_1 - \mathbf{L}_1) + (\hat{K}_{BA} - K_{BA})(\hat{\mathbf{L}}_1 - \mathbf{L}_1).
$$

Considering a probability bound for each of these three subterms, we have

$$
\begin{aligned}
&\mathbb{P}\{\|(\hat{K}_{BA} - K_{BA})\mathbf{L}_1\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|\hat{K}_{BA} - K_{BA}\|_2 \|\mathbf{L}_1\|_2 \geq \varepsilon\} && \text{(by submultiplicativity)} \\
&= \mathbb{P}\left\{\|\hat{K}_{BA} - K_{BA}\|_2 \geq \frac{\varepsilon}{\|\mathbf{L}_1\|_2}\right\} \\
&\leq \eta_{AB}\left(\frac{\varepsilon}{\|\mathbf{L}_1\|_2}\right), && \text{(by (G.10))}
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{P}\{\|K_{BA}(\hat{\mathbf{L}}_1 - \mathbf{L}_1)\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|K_{BA}\|_2 \|\hat{\mathbf{L}}_1 - \mathbf{L}_1\|_2 \geq \varepsilon\} && \text{(by submultiplicativity)} \\
&\leq \mathbb{P}\{\|A\|\|B\|\|\hat{\mathbf{L}}_1 - \mathbf{L}_1\|_2 \geq \varepsilon\} && (\text{since } \|K_{BA}\|_2 = \|P_1(B \otimes A)\|_2 \leq \|P_1\|_2 \|B \otimes A\|_2 = \|A\|_2 \|B\|_2) \\
&= \mathbb{P}\left\{\|\hat{\mathbf{L}}_1 - \mathbf{L}_1\|_2 \geq \frac{\varepsilon}{\|A\|_2 \|B\|_2}\right\} \\
&\leq \eta_L\left(\frac{\varepsilon}{\|A\|_2 \|B\|_2}\right), && \text{(by (G.4))}
\end{aligned}
$$

and

$$
\begin{aligned}
&\mathbb{P}\{\|(\hat{K}_{BA} - K_{BA})(\hat{\mathbf{L}}_1 - \mathbf{L}_1)\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|\hat{K}_{BA} - K_{BA}\|_2 \geq \sqrt{\varepsilon}\} + \mathbb{P}\{\|\hat{\mathbf{L}}_1 - \mathbf{L}_1\|_2 \geq \sqrt{\varepsilon}\} && \text{(by (E.5))} \\
&\leq \eta_{AB}\left(\sqrt{\varepsilon}\right) + \eta_L\left(\sqrt{\varepsilon}\right). && \text{(by (G.10) and (G.4))}
\end{aligned}
$$

Putting together the bounds for the three subterms,

$$
\begin{aligned}
&\mathbb{P}\{\|\hat{K}_{BA}\hat{\mathbf{L}}_1 - K_{BA}\mathbf{L}_1\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|(\hat{K}_{BA} - K_{BA})\mathbf{L}_1\|_2 \geq \varepsilon/3\} + \mathbb{P}\{\|K_{BA}(\hat{\mathbf{L}}_1 - \mathbf{L}_1)\|_2 \geq \varepsilon/3\} + \mathbb{P}\{\|(\hat{K}_{BA} - K_{BA})(\hat{\mathbf{L}}_1 - \mathbf{L}_1)\|_2 \geq \varepsilon/3\} \\
&\hfill \text{(by (E.4))} \\
&\leq \eta_{AB}\left(\frac{\varepsilon}{3\,\|\mathbf{L}_1\|_2}\right) + \eta_L\left(\frac{\varepsilon}{3\|A\|_2\|B\|_2}\right) + \eta_{AB}\left(\sqrt{\varepsilon/3}\right) + \eta_L\left(\sqrt{\varepsilon/3}\right) \\
&=: \eta_{KL}(\varepsilon).
\end{aligned}
$$

For the fourth term, $\hat{K}_{AB}\hat{\mathbf{L}}_2 - K_{AB}\mathbf{L}_2$, an identical argument to that for the third term using (G.10) and (G.5) yields

$$
\mathbb{P}\{\|\hat{K}_{AB}\hat{\mathbf{L}}_2 - K_{AB}\mathbf{L}_2\|_2 \geq \varepsilon\} \leq \eta_{KL}(\varepsilon).
$$

For the fifth term, $(\hat{\tilde{B}} - \tilde{B})\mathbf{U}$, we have

$$
\begin{aligned}
&\mathbb{P}\{\|(\hat{\tilde{B}} - \tilde{B})\mathbf{U}\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|\hat{\tilde{B}} - \tilde{B}\|_2 \|\mathbf{U}\|_2 \geq \varepsilon\} && \text{(by submultiplicativity)} \\
&= \mathbb{P}\left\{\|\hat{\tilde{B}} - \tilde{B}\|_2 \geq \frac{\varepsilon}{\|\mathbf{U}\|_2}\right\} \\
&\leq \eta_B\left(\frac{\varepsilon}{\|\mathbf{U}\|_2}\right). && \text{(by (G.8))}
\end{aligned}
$$

Putting together the bounds for the five terms, we have

$$
\begin{aligned}
&\mathbb{P}\{\|\hat{\mathbf{C}} - \mathbf{C}\|_2 \geq \varepsilon\} \\
&\leq \mathbb{P}\{\|\hat{\mathbf{M}}_2 - \mathbf{M}_2\|_2 \geq \varepsilon/5\} + \mathbb{P}\{\|\hat{\tilde{A}}\hat{\mathbf{M}}_1 - \tilde{A}\mathbf{M}_1\|_2 \geq \varepsilon/5\} + \mathbb{P}\{\|\hat{K}_{BA}\hat{\mathbf{L}}_1 - K_{BA}\mathbf{L}_1\|_2 \geq \varepsilon/5\} \\
&\quad + \mathbb{P}\{\|\hat{K}_{AB}\hat{\mathbf{L}}_2 - K_{AB}\mathbf{L}_2\|_2 \geq \varepsilon/5\} + \mathbb{P}\{\|(\hat{\tilde{B}} - \tilde{B})\mathbf{U}\|_2 \geq \varepsilon/5\} \qquad \text{(by (E.4))} \\
&\leq \eta_D(\varepsilon/5) + \eta_{AM}(\varepsilon/5) + 2\eta_{KL}(\varepsilon/5) + \eta_B\left(\frac{\varepsilon}{5\|\mathbf{U}\|}\right) \\
&=: \eta_C(\varepsilon).
\end{aligned}
$$

Lemma 9 *Suppose Assumptions 1 and 2 hold. Then for all $\varepsilon > 0$,*

$$\mathbb{P}\{\|\hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal - \mathbf{C}\mathbf{D}^\intercal\|_2 \geq \varepsilon\} \leq \eta_{CD}(\varepsilon),$$

where

$$\eta_{CD}(\varepsilon) := \eta_C\left(\sqrt{\frac{\varepsilon}{3}}\right) + \eta_D\left(\sqrt{\frac{\varepsilon}{3}}\right) + \eta_C\left(\frac{\varepsilon}{3\|\mathbf{D}\|_2}\right) + \eta_D\left(\frac{\varepsilon}{3\|\mathbf{C}\|_2}\right).$$

PROOF. The proof follows from using the decomposition

$$\hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal - \mathbf{C}\mathbf{D}^\intercal = (\hat{\mathbf{C}} - \mathbf{C})(\hat{\mathbf{D}} - \mathbf{D})^\intercal + (\hat{\mathbf{C}} - \mathbf{C})\mathbf{D}^\intercal + \mathbf{C}(\hat{\mathbf{D}} - \mathbf{D})^\intercal$$

to provide conservative decompositions into terms of the form

$$\mathbb{P}\{\|\hat{\mathbf{C}} - \mathbf{C}\|_2 \geq \varepsilon\} \leq \eta,\ \mathbb{P}\{\|\hat{\mathbf{D}} - \mathbf{D}\|_2 \geq \varepsilon\} \leq \eta,$$

which are suitable for the bounds of Lemma 8.

Lemma 10 *Suppose Assumptions 1 and 2 hold. Given a positive value $\varepsilon_{\max}$, then for all $0 < \varepsilon < \varepsilon_{\max}$,*

$$\mathbb{P}\{\|(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^\dagger - (\mathbf{D}\mathbf{D}^\intercal)^{-1}\|_2 \geq \varepsilon\} \leq \eta_{DD}(\varepsilon, \varepsilon_{\max}),$$

where

$$
\begin{aligned}
\eta_{DD}(\varepsilon, \varepsilon_{\max}) &:= \eta_0\left(\frac{1}{2}\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)\left(1 - \frac{\varepsilon}{\varepsilon_{\max}}\right)\varepsilon\right) + \eta_m\left(\frac{\varepsilon\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)/\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal))}\right), \\
\eta_0(\varepsilon) &:= \eta_D\left(\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal) + \varepsilon} - \sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\right), \\
\eta_m(\varepsilon) &:= \left(9^{[n(n+1)+m(m+1)]/2} + \left(\frac{16\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)} + 1\right)^{[n(n+1)+m(m+1)]/2}\right)\eta_0(\varepsilon).
\end{aligned}
$$

PROOF. The proof follows an identical argument to Lemma 7:

(1) Replace $\mathbf{Z}$ by $\mathbf{D}$, n by $n(n+1)/2$, and m by $m(m+1)/2$.
(2) we apply (E.2) to obtain the decomposition

$$(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^\dagger - (\mathbf{D}\mathbf{D}^\intercal)^{-1} = (\mathbf{D}\mathbf{D}^\intercal)^{-1}(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^\dagger\left[(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal) - (\mathbf{D}\mathbf{D}^\intercal)\right].$$

(3) Apply Lemma 8 with the appropriate settings of ε to get the bound

$$\mathbb{P}\{\|\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal - \mathbf{D}\mathbf{D}^\intercal\|_2 \geq \varepsilon\} \leq \eta_D\left(\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal) + \varepsilon} - \sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\right) =: \eta_0(\varepsilon). \tag{G.12}$$

(4) Apply a similar γ-net argument to obtain an upper bound of $\lambda_{\max}(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)$, then a lower bound of $\lambda_{\min}(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)$, and finally the claimed bound.

Theorem 5 (Theorem 3 restated) *Suppose Assumptions 1 and 2 hold. Given a positive value $\varepsilon_{\max}$, then for all* $0 < \varepsilon < 3\varepsilon_{\max} \cdot \min\{\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}, \varepsilon_{\max}\}$,

$$\mathbb{P}\left\{\left\|\left[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B\right] - \left[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B\right]\right\|_2 \geq \varepsilon\right\} \leq \eta(\varepsilon),$$

where

$$\eta(\varepsilon) := \eta_{CD}\left(\frac{1}{3}\lambda_{\min}(\mathbf{C}\mathbf{C}^\intercal)\varepsilon\right) + \eta_{CD}\left(\sqrt{\frac{\varepsilon}{3}}\right) + \eta_{DD}\left(\frac{\varepsilon}{3\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}}, \varepsilon_{\max}\right) + \eta_{DD}\left(\sqrt{\frac{\varepsilon}{3}}, \varepsilon_{\max}\right).$$

PROOF. The proof follows an identical argument to Theorem 4:

(1) Replace A and B by $\tilde{\Sigma}'_A$ and $\tilde{\Sigma}'_B$, and replace $\mathbf{Y}$ and $\mathbf{Z}$ by $\mathbf{C}$ and $\mathbf{D}$.
(2) Decompose the error matrix using the least-squares estimators as

$$\begin{aligned}
&\left[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B\right] - \left[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B\right] \\
&= \hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^{-1} - \mathbf{C}\mathbf{D}^\intercal(\mathbf{D}\mathbf{D}^\intercal)^{-1} \\
&= \left[\hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal - \mathbf{C}\mathbf{D}^\intercal\right](\mathbf{D}\mathbf{D}^\intercal)^{-1} + \mathbf{C}\mathbf{D}^\intercal\left[(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^{-1} - (\mathbf{D}\mathbf{D}^\intercal)^{-1}\right] + \left[\hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal - \mathbf{C}\mathbf{D}^\intercal\right]\left[(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^{-1} - (\mathbf{D}\mathbf{D}^\intercal)^{-1}\right].
\end{aligned}$$

(3) Consider a probability bound and use (E.4).
(4) Apply Lemmas 9 and 10 with the appropriate settings of ε in each term.
(5) The conclusion follows by combining the probability bounds for each term.

PROOF OF THEOREM 3.
The qualitative claim in Theorem 3 is found by inverting the bound of Theorem 5 and examining the behavior of the bound as $n_r \to \infty$. The argument is similar to the proof of Theorem 2, so we just state the major steps.

From Lemma 8, it follows that for fixed $\delta \in (0, 1)$ and $\varepsilon_D(\delta) > 0$ such that $\mathbb{P}\{\|\hat{\mathbf{D}} - \mathbf{D}\|_2 \geq \varepsilon_D(\delta)\} \leq \eta_D(\varepsilon_D(\delta)) = \delta$,

$$\varepsilon_D(\delta) = \mathcal{O}\left(\sqrt{\frac{\ell c_F^2 \log\{[n(n+1)/2 + \ell]/\delta\}}{n_r}}\right).$$

Write $\eta_C(\varepsilon)$ in Lemma 8 explicitly,

$$\begin{aligned}
\eta_C(\varepsilon) = {} & \eta_D(\varepsilon/5) + \eta_D\left(\frac{\varepsilon}{30\|A\|_2^2}\right) + \eta_D(\sqrt{\varepsilon/15}) + 2\eta_L\left(\frac{\varepsilon}{15\|A\|_2\|B\|_2}\right) + 2\eta_L\left(\sqrt{\varepsilon/15}\right) + \delta_{AB}\left(\frac{1}{2}\sqrt{\frac{\varepsilon}{15\|\mathbf{M}_1\|_2}}\right) \\
& + \delta_{AB}\left(\frac{\varepsilon}{120\|\mathbf{M}_1\|_2\sqrt{\|A\|_2}}\right) + \delta_{AB}\left(\frac{1}{2}\left(\frac{\varepsilon}{15}\right)^{\frac{1}{4}}\right) + \delta_{AB}\left(\frac{1}{8}\sqrt{\frac{\varepsilon}{15\|A\|_2}}\right) + \delta_{AB}\left(\frac{1}{2}\sqrt{\frac{\varepsilon}{5\|\mathbf{U}\|_2}}\right) \\
& + \delta_{AB}\left(\frac{\varepsilon}{40\|\mathbf{U}\|_2\sqrt{\|B\|_2}}\right) + 4\delta_{AB}\left(\sqrt{\frac{\varepsilon}{45\|\mathbf{L}_1\|_2}}\right) + 2\delta_{AB}\left(\frac{\varepsilon}{45\|\mathbf{L}_1\|_2\sqrt{\|A\|_2}}\right) + 2\delta_{AB}\left(\frac{\varepsilon}{45\|\mathbf{L}_1\|_2\sqrt{\|B\|_2}}\right) \\
& + 4\delta_{AB}\left(\frac{1}{\sqrt{3}}\left(\frac{\varepsilon}{15}\right)^{\frac{1}{4}}\right) + 2\delta_{AB}\left(\frac{1}{3\sqrt{\|A\|_2}}\sqrt{\frac{\varepsilon}{15}}\right) + 2\delta_{AB}\left(\frac{1}{3\sqrt{\|B\|_2}}\sqrt{\frac{\varepsilon}{15}}\right).
\end{aligned}$$

Hence for fixed $\delta \in (0,1)$ we can find $\varepsilon_C(\delta) > 0$ such that $\mathbb{P}\{\|\hat{\mathbf{C}} - \mathbf{C}\|_2 \geq \varepsilon_C(\delta)\} \leq \eta_C(\varepsilon_C(\delta)) \leq \delta$ holds for large enough n_r, $\varepsilon_L(\delta)$ such that $\eta_L(\varepsilon_L(\delta)) = \delta$, and $\varepsilon_{AB}(\delta)$, given in (F.12)+. That is,

$$
\begin{aligned}
&\varepsilon_C(\delta)\\
&= \mathcal{O}\Big(\max\Big\{5\varepsilon_D(\delta/17), 30\|A\|_2^2\varepsilon_D(\delta/17), 15\varepsilon_D^2(\delta/17), 15\|A\|_2\|B\|_2\varepsilon_L(\delta/34), 15\varepsilon_L^2(\delta/34), 60\|\mathbf{M_1}\|_2\varepsilon_{AB}^2(\delta/17),\\
&\quad 120\|\mathbf{M_1}\|_2\sqrt{\|A\|_2}\varepsilon_{AB}(\delta/17), 240\varepsilon_{AB}^4(\delta/17), 960\|A\|_2\varepsilon_{AB}^2(\delta/17), 20\|\mathbf{U}\|_2\varepsilon_{AB}^2(\delta/17), 40\|\mathbf{U}\|_2\sqrt{\|B\|_2}\varepsilon_{AB}(\delta/17),\\
&\quad 45\|\mathbf{L}_1\|_2\varepsilon_{AB}^2(\delta/68), 45\|\mathbf{L}_1\|_2\sqrt{\|A\|_2}\varepsilon_{AB}(\delta/34), 45\|\mathbf{L}_1\|_2\sqrt{\|B\|_2}\varepsilon_{AB}(\delta/34), 135\varepsilon_{AB}^4(\delta/68), 135\|A\|_2\varepsilon_{AB}^2(\delta/34),\\
&\quad 135\|A\|_2\varepsilon_{AB}^2(\delta/34)\Big\}\Big)\\
&= \mathcal{O}\big(\max\big\{5\varepsilon_D(\delta/17), 30\|A\|_2^2\varepsilon_D(\delta/17), 15\|A\|_2\|B\|_2\varepsilon_L(\delta/34), 120\|\mathbf{M_1}\|_2\sqrt{\|A\|_2}\varepsilon_{AB}(\delta/17),\\
&\quad 40\|\mathbf{U}\|_2\sqrt{\|B\|_2}\varepsilon_{AB}(\delta/17), 45\|\mathbf{L}_1\|_2\sqrt{\|A\|_2}\varepsilon_{AB}(\delta/34), 45\|\mathbf{L}_1\|_2\sqrt{\|B\|_2}\varepsilon_{AB}(\delta/34)\big\}\big)\\
&= \mathcal{O}\left(\max\left\{ \|A\|_2^2\sqrt{\frac{17\ell c_F^2\log\{[n(n+1)/2+\ell]/\delta\}}{n_r}}, \|A\|_2\|B\|_2\sqrt{\frac{34\ell c_W^2\log[(nm+\ell)/\delta]}{n_r}},\right.\right.\\
&\quad \left.\left. \max\{\|\mathbf{M_1}\|_2\sqrt{\|A\|_2}, \|\mathbf{U}\|_2\sqrt{\|B\|_2}, \|\mathbf{L}_1\|_2\sqrt{\|A\|_2}, \|\mathbf{L}_1\|_2\sqrt{\|B\|_2}\}\varepsilon_{AB}(\delta/34)\right\}\right),
\end{aligned}
$$

where in the last equation we drop the constants and only show the dependence of the bound on system parameters, and $\varepsilon_{AB}(\delta)$ is given in (F.12).

Next, for $\delta \in (0,1)$, let

$$
\begin{aligned}
\varepsilon_{CD}(\delta) &= \max\{3\varepsilon_C^2(\delta/4), 3\varepsilon_D^2(\delta/4), 3\|\mathbf{D}\|_2\varepsilon_C(\delta/4), 3\|\mathbf{C}\|_2\varepsilon_D(\delta/4)\}\\
&= \mathcal{O}\left(\max\{\|\mathbf{D}\|_2\varepsilon_C(\delta/4), \|\mathbf{C}\|_2\varepsilon_D(\delta/4)\}\right),
\end{aligned}
$$

and then from Lemma 9 we know that

$$
\mathbb{P}\big\{\big\|\hat{\mathbf{C}}\hat{\mathbf{D}}^\intercal - \mathbf{C}\mathbf{D}^\intercal\big\|_2 \geq \varepsilon_{CD}(\delta)\big\} \leq \eta_{CD}(\varepsilon_{CD}(\delta)) \leq \delta.
$$

Under the condition of Lemma 10, for fixed $\delta \in (0,1)$, define $\varepsilon_{\eta 0}(\delta), \varepsilon_{\eta m}(\delta) > 0$ as follows such that $\delta_0(\varepsilon_{\eta 0}(\delta)) = \delta$ and $\delta_m(\varepsilon_{\eta m}(\delta)) = \delta$

$$
\begin{aligned}
\varepsilon_{\eta 0}(\delta) &:= \varepsilon_D^2(\delta) + 2\varepsilon_D(\delta)\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)} = \mathcal{O}\left(\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_D(\delta)\right),\\
\varepsilon_{\eta m}(\delta) &:= \varepsilon_D^2(\delta/d(n,m)) + 2\varepsilon_D(\delta/d(n,m))\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)} = \mathcal{O}\left(\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_D(\delta/d(n,m))\right),
\end{aligned}
$$

where

$$
d(n,m) := 9^{[n(n+1)+m(m+1)]/2} + \left(\frac{16\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)} + 1\right)^{[n(n+1)+m(m+1)]/2}. \tag{G.13}
$$

For fixed $\delta, \varepsilon_{\max} \in (0,1)$, let $\varepsilon_{D1}(\delta) \in (0, \varepsilon_{\max})$ such that

$$
\varepsilon_{D1}(\delta) = \frac{1}{2}\varepsilon_{\max}\left(1 - \sqrt{1 - \frac{8\varepsilon_{\eta 0}(\delta)}{\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)}}\right) = \mathcal{O}\left(\frac{\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}}{\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_D(\delta)\right),
$$

and set $\varepsilon_{D2}(\delta) \in (0, \varepsilon_{\max})$ such that

$$
\varepsilon_{D2}(\delta) = \frac{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)/\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal))}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_{\eta m}(\delta)
$$

$$= \mathcal{O}\left(\left(2 + \frac{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\right)\frac{\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_D(\delta/d(n,m))\right).$$

Now define $\varepsilon_{DD}(\delta) := \max\{\varepsilon_{D1}(\delta/2), \varepsilon_{D2}(\delta/2)\}$. Since for fixed $\delta, \varepsilon_{\max} \in (0,1)$, when n_r is large enough, $\varepsilon_{D1}(\delta/2)$, $\varepsilon_{D2}(\delta/2) < \varepsilon_{\max}/2$, it holds that

$$\begin{aligned}
&\mathbb{P}\big\{\big\|(\hat{\mathbf{D}}\hat{\mathbf{D}}^\intercal)^\dagger - (\mathbf{D}\mathbf{D}^\intercal)^{-1}\big\|_2 \geq \varepsilon_{DD}(\delta)\big\} \\
&\leq \eta_{DD}(\varepsilon_{DD}(\delta), \varepsilon_{\max}) \\
&= \eta_0\left(\frac{1}{2}\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)\left(1 - \frac{\varepsilon_{DD}(\delta)}{\varepsilon_{\max}}\right)\varepsilon_{DD}(\delta)\right) + \eta_m\left(\frac{\varepsilon_{DD}(\delta)\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)/\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal))}\right) \\
&\leq \eta_0\left(\frac{1}{2}\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)\left(1 - \frac{\varepsilon_{D1}(\delta/2)}{\varepsilon_{\max}}\right)\varepsilon_{D1}(\delta/2)\right) + \eta_m\left(\frac{\varepsilon_{D2}(\delta/2)\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}{\varepsilon_{\max}(2 + \lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)/\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal))}\right) = \delta.
\end{aligned}$$

Finally, let

$$\varepsilon_\Sigma(\delta) := \max\left\{\frac{3\varepsilon_{CD}(\delta/4)}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}, 3\varepsilon_{CD}^2(\delta/4), 3\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_{DD}(\delta/4), 3\varepsilon_{DD}^2(\delta/4)\right\},$$

and it can be observed that for fixed $\delta \in (0,1)$ and large enough n_r,

$$\begin{aligned}
&\mathbb{P}\left\{\left\|\left[\hat{\tilde{\Sigma}}'_A\ \hat{\tilde{\Sigma}}'_B\right] - \left[\tilde{\Sigma}'_A\ \tilde{\Sigma}'_B\right]\right\|_2 \geq \varepsilon_\Sigma(\delta)\right\} \\
&\leq \eta(\varepsilon_\Sigma(\delta)) \\
&= \eta_{CD}\left(\frac{1}{3}\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)\varepsilon_\Sigma(\delta)\right) + \eta_{CD}\left(\sqrt{\frac{\varepsilon_\Sigma(\delta)}{3}}\right) + \eta_{DD}\left(\frac{\varepsilon_\Sigma(\delta)}{3\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}}, \varepsilon_{\max}\right) \\
&\quad + \eta_{DD}\left(\sqrt{\frac{\varepsilon_\Sigma(\delta)}{3}}, \varepsilon_{\max}\right) \\
&\leq \eta_{CD}(\varepsilon_{CD}(\delta/4)) + \eta_{CD}(\varepsilon_{CD}(\delta/4)) + \eta_{DD}(\varepsilon_{DD}(\delta/4), \varepsilon_{\max}) + \eta_{DD}(\varepsilon_{DD}(\delta/4), \varepsilon_{\max}) = \delta.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\varepsilon_\Sigma(\delta) \\
&= \mathcal{O}\Bigg(\max\left\{\frac{\varepsilon_{CD}(\delta/4)}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}, \sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_{DD}(\delta/4)\right\}\Bigg) \\
&= \mathcal{O}\Bigg(\max\Bigg\{\frac{\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_C(\delta/4), \frac{\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)}}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_D(\delta/4), \frac{\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)}\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_D(\delta/8), \\
&\quad \sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)}\left(1 + \frac{2\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\right)\varepsilon_D(\delta/8d(n,m))\Bigg\}\Bigg) \\
&= \mathcal{O}\Bigg(\max\Bigg\{\frac{\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\|A\|_2^2}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\sqrt{\frac{17\ell c_F^2\log\{4[n(n+1)/2+\ell]/\delta\}}{n_r}}, \frac{\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)}}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\sqrt{\frac{\ell c_F^2\log\{4[n(n+1)/2+\ell]/\delta\}}{n_r}}, \\
&\quad \frac{\sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)}\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\min}^2(\mathbf{D}\mathbf{D}^\intercal)}\sqrt{\frac{\ell c_F^2\log\{8[n(n+1)/2+\ell]/\delta\}}{n_r}}, \\
&\quad \sqrt{\lambda_{\max}(\mathbf{C}\mathbf{C}^\intercal)}\left(1 + \frac{2\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\right)\sqrt{\frac{\ell c_F^2\log\{8[n(n+1)/2+\ell]d(n,m)/\delta\}}{n_r}} \\
&\quad \frac{\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}\|A\|_2\|B\|_2}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\sqrt{\frac{34\ell c_W^2\log[4(nm+\ell)/\delta]}{n_r}},
\end{aligned}$$

$$\max\{\|\mathbf{M_1}\|_2\sqrt{\|A\|_2}, \|\mathbf{U}\|_2\sqrt{\|B\|_2}, \|\mathbf{L}_1\|_2\sqrt{\|A\|_2}, \|\mathbf{L}_1\|_2\sqrt{\|B\|_2}\}\frac{\sqrt{\lambda_{\max}(\mathbf{D}\mathbf{D}^\intercal)}}{\lambda_{\min}(\mathbf{D}\mathbf{D}^\intercal)}\varepsilon_{AB}(\delta/136)\Bigg\}\Bigg),$$

where $\varepsilon_{AB}(\delta)$ is given in (F.12), and $d(n,m)$ is given in (G.13). This completes the proof by noticing the bound of $\varepsilon_{AB}(\delta)$ in the proof of Theorem 2.