

Exact Inference: Variable Elimination

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Sum-product Inference Problems

- ▶ Given a PGM containing factors ϕ_1, \dots, ϕ_m and an assignment $\mathbf{E} = \mathbf{e}$ to a subset of its variables (evidence), find
 1. $\Pr(\mathbf{E} = \mathbf{e})$: Probability of evidence
 2. $\Pr(X|\mathbf{E} = \mathbf{e})$: Conditional distribution at a variable X given evidence (marginals)

By definition, these are sum-product tasks and can be reduced to computing partition function of a Markov network. Why?

$$Z = \sum_{\mathbf{x}} \prod_{i=1}^m \phi_i(\mathbf{x}_i)$$

where \mathbf{x}_i is the projection of \mathbf{x} on the variables involved in ϕ_i .

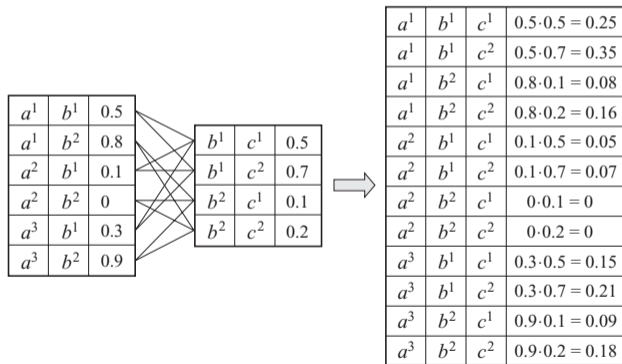
Naive sum-product algorithm

$$Z = \sum_{\mathbf{x}} \prod_{i=1}^m \phi_i(\mathbf{x}_i)$$

- ▶ Initialize $Z = 0$
- ▶ Iterate over all possible assignments \mathbf{x}
 - ▶ Compute the value of each assignment $value(\mathbf{x}) = \prod_{i=1}^m \phi_i(\mathbf{x}_i)$ (Product)
 - ▶ $Z = Z + value(\mathbf{x})$ (Sum)
- ▶ **Return** Z

Product Operation

$$\phi(A, B, C) = \phi(A, B) \times \phi(B, C)$$



Sum-out Operation

$$\sum_{a \in A} \phi(A, B) = \psi(B)$$

$$\sum_{a \in A} \begin{array}{|c|c|c|} \hline & \phi(A, B) & \\ \hline a^0 & b^0 & 30 \\ a^0 & b^1 & 5 \\ a^1 & b^0 & 1 \\ a^1 & b^1 & 10 \\ \hline \end{array} = \begin{array}{|c|c|} \hline & \psi(B) \\ \hline b^0 & \\ b^1 & \\ \hline \end{array}$$

Idea: Distribute Sums over Products

Example:

$\phi(A, B)$		
a^0	b^0	30 (w_0)
a^0	b^1	5 (w_1)
a^1	b^0	1 (w_2)
a^1	b^1	10 (w_3)

$\phi(B, C)$		
b^0	c^0	100 (w_4)
b^0	c^1	1 (w_5)
b^1	c^0	1 (w_6)
b^1	c^1	100 (w_7)

$$Z = \sum_{b \in B} \sum_{a \in A} \sum_{c \in C} \phi(a, b) \phi(b, c) \quad (1)$$

$$= \sum_{b \in B} \sum_{a \in A} \phi(a, b) \left\{ \sum_{c \in C} \phi(b, c) \right\} \quad (2)$$

$$= \sum_{b \in B} \psi(b) \left\{ \sum_{a \in A} \phi(a, b) \right\} \quad (3)$$

$$= \sum_{b \in B} \psi(b) \chi(b) \quad (4)$$

Idea: Distribute Sums over Products

Example:

$\phi(A, B)$

a^0	b^0	$30 (w_0)$
a^0	b^1	$5 (w_1)$
a^1	b^0	$1 (w_2)$
a^1	b^1	$10 (w_3)$

$\phi(B, C)$

b^0	c^0	$100 (w_4)$
b^0	c^1	$1 (w_5)$
b^1	c^0	$1 (w_6)$
b^1	c^1	$100 (w_7)$

$$\begin{aligned} Z &= w_0 w_4 + w_0 w_5 + w_1 w_6 + w_1 w_7 \\ &\quad + w_2 w_4 + w_2 w_5 + w_3 w_6 + w_3 w_7 \\ &= w_0(w_4 + w_5) + w_1(w_6 + w_7) \\ &\quad + w_2(w_4 + w_5) + w_3(w_6 + w_7) \\ &= (w_0 + w_2)(w_4 + w_5) \\ &\quad + (w_1 + w_3)(w_6 + w_7) \end{aligned}$$

Bucket Elimination

Intuitive data-structure for performing variable elimination!

- ▶ Let X_1, \dots, X_n be an ordering of the variables and let Φ denote the current set (database) of functions
- ▶ Associate each variable with a bucket
- ▶ Process the buckets from top to bottom X_1, \dots, X_n
 - ▶ Put all functions in Φ that mention X_i in the bucket of X_i and remove them from Φ
 - ▶ Compute the product of all functions in the bucket of X_i . Let us call the resulting function ψ
 - ▶ Sum-out X_i from ψ . Call the new function χ
 - ▶ Add χ to Φ

Important: At the end, Φ will contain one or more functions with one entry. The partition function equals the product of these entries.

Bucket Elimination: Example and Animation