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Supplementary Material

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In this supplementary material, we provide a proof for Eq. 7 in our manuscript. Note this proof is the discrete analogy of Sec. 3 in [RA15].

Lemma: For a fixed cluster $\mathcal{C}$, the minimization of our defined color homogeneity, which is

$$E_{color}(\mathcal{C}) = \min_{\bar{\mathbf{c}}, \Delta=\Delta^\top, |\Delta|=1} \sum_{\mathbf{x}\in\mathcal{C}} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}})^\top \Delta^{-1} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}}).$$

can be obtained by the optimal observation point $\bar{\mathbf{c}}^\star$ and the optimal variation matrix$(\Delta^\star)^{-1}$ in Eq. 7 of our manuscript:

$$\bar{\mathbf{c}}^\star = \sum_{\mathbf{x}\in\mathcal{C}} \mathcal{I}(\mathbf{x}) / |\mathcal{C}|,$$

$$(\Delta^\star)^{-1} = |\mathbf{U}_c(\mathcal{C})|^{\frac{1}{d}} \mathbf{U}_c^{-1}(\mathcal{C}).$$

Proof: Note we have assumed Δ is symmetry and positive definite. For this constrained optimization problem, we consider the Lagrangian:

$$L = \sum_{\mathbf{x}\in\mathcal{C}} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}})^\top \Delta^{-1} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}}) + \mu(\left|\Delta^{-1}\right| - 1).$$

The optimal observation point $\bar{\mathbf{c}}^\star$ can be obtained by:

$$\mathbf{0} = \frac{\partial L}{\partial \bar{\mathbf{c}}}|_{\bar{\mathbf{c}}=\bar{\mathbf{c}}^\star} = \Delta^{-1} \sum_{\mathbf{x}\in\mathcal{C}} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}}^\star).$$

Thus optimal observation point is the mean value of the observations regardless of the variation matrix:

$$\bar{\mathbf{c}}^\star = \sum_{\mathbf{x}\in\mathcal{C}} \mathcal{I}(\mathbf{x}) / |\mathcal{C}|.$$

The optimal variation matrix$(\Delta^\star)^{-1}$ can be obtained by the same approach. From [PP12], we note that:

$$\frac{\partial \sum_{\mathbf{x}\in\mathcal{C}} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}})^\top \Delta^{-1} (\mathcal{I}(\mathbf{x}) - \bar{\mathbf{c}})}{\partial \Delta^{-1}} = \mathbf{U}_c(\mathcal{C}),$$

$$\frac{\partial \left|\Delta^{-1}\right|}{\partial \Delta^{-1}} = \left|\Delta^{-1}\right| \Delta^\top.$$

Let us write the condition for the optimal variation matrix$(\Delta^\star)^{-1}$:

$$\mathbf{0} = \frac{\partial L}{\partial (\Delta)^{-1}}|_{(\Delta)^{-1}=(\Delta^\star)^{-1}} = \mathbf{U}_c(\mathcal{C}) + \mu \left|(\Delta^\star)^{-1}\right| (\Delta^\star)^\top. \quad (1)$$

From Eq. 1, we can clearly see that the optimal variation matrix $\Delta^\star$ is equal to $\mathbf{U}_c(\mathcal{C})$ up to scale, where μ guarantees the unity determinant constraint. Thus:

$$(\Delta^\star)^{-1} = |\mathbf{U}_c(\mathcal{C})|^{\frac{1}{d}} \mathbf{U}_c^{-1}(\mathcal{C}). \qquad \square$$

References

[PP12] PETERSEN K. B., PEDERSEN M. S.: The matrix cookbook, 2012. 1

[RA15] RICHTER R., ALEXA M.: Mahalanobis centroidal Voronoi tessellations. *Computers & Graphics 46*, 0 (2015), 48–54. 1