Performance Robustness Analysis of VLSI Circuits with Process Variations Based on Kharitonov's Theorem*

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***Abstract*— In today's VLSI technology, the process variations are unavoidable. This paper proposes an efficient analysis approach for exploring the worst case performance for VLSI circuits with severe parameter value variations due to nano-scale process. Inspired by Kharitonov's theorem, the described method dramatically reduces the computational burden to only evaluate several critical Kharitonov-type interval transfer functions. The computational efficiency of the method is demonstrated by two practical VLSI circuits.**

I. Introduction

As the advanced VLSI technology has reached 22nm feature size and below, process lithography no longer produces the ideal dimension of circuit components. The corresponding electrical parameters may vary as large as 1/3 or more [1]. Therefore, today's computer-aided design (CAD) tools are not only applied to design nominal VLSI circuits to meet the specs, but also assist in performance robustness analysis of nano-scale VLSI circuits with the effect from the unavoidable manufactured fluctuations. Current CAD tools to deal with performance robustness analysis of VLSI circuits with process variations are mainly developed along the line of the Monte Carlo method or similar stochastic and statistical analysis methods, e.g., [2], [3]. However, they all require long simulation time and underestimate the likely performance variation range when the sampling numbers are not sufficient enough. A main concern is to determine the worst case performance range of nano-scale VLSI circuits without evaluation of large number of samples as required by Monte Carlo method.

In this paper, we interest in calculating the worst case frequency-domain performance of VLSI circuits with manufactured variations. The uncertainties of electrical parameters are represented as real interval numbers. Given an uncertain electrical parameter p, suppose there is a variation level of ±30% to it, then p varies within an interval of $[p^-, p^+]$. p^- and p^+ are the endpoints of the interval, and p^- equals to $p\times(1\text{-}30\%)$, while p^+ equals to $p\times(1+30\%)$. Note that it is not necessary to know the statistical properties of p but the endpoints of the interval. Consider the behavior of a VLSI circuit is governed by an interval transfer function:

$$H(s;a,b) = \frac{N(s,a)}{D(s,b)} = \frac{a_0 + a_1 s + \ldots + a_m s^m}{b_0 + b_1 s + \ldots + b_n s^n} \qquad (1)$$

where a_i $(i=0,1,\ldots,m)$ and b_i $(i=0,1,\ldots n)$ are uncertain coefficients for numerator and denominator. They vary randomly within $[a_i^-, a_i^+]$ and $[b_i^-, b_i^+]$ respectively. In practical VLSI circuits, a_i and b_i are usually constituted of several underlying parameters. Therefore, the variation ranges of coefficients should be converted from their corresponding parameters' variation ranges by applying interval arithmetic [4].

The rest of this paper is organized as follows. Section II briefly explains the geometrical properties of Kharitonov's theorem. Section III introduces critical transfer functions and their construction. The application of the proposed idea to two practical VLSI circuits is described in section IV. Finally, section V concludes this paper.

II. Kharitonov's Theorem and its Geometrical Properties

Kharitonov's Theorem [5] is a seminal theorem to determine robust stability of perturbed control systems. Given a perturbed control system which is modeled by a family of real interval polynomials:

$$p(s) = q_0 + q_1 s + \ldots + q_m s^m, q \in Q \qquad (2)$$

where q_i is uncertain coefficient with variation range $[q_i^-, q_i^+]$. Kharitonov's theorem states that the Hurwitz stability of a family of real interval polynomials can be guaranteed by the Hurwitz stability of four prescribed critical vertex polynomials (Kharitonov's Polynomials) in this family which are given as follows:

$$K_1(s) = P_{el}(s) + P_{ol}(s) = (q_0^- + q_2^+ s^2 + \ldots) + (q_1^- s + q_3^+ s^3 + \ldots) \qquad (3a)$$

$$K_2(s) = P_{el}(s) + P_{oh}(s) = (q_0^- + q_2^+ s^2 + \ldots) + (q_1^+ s + q_3^- s^3 + \ldots) \qquad (3b)$$

$$K_3(s) = P_{eh}(s) + P_{ol}(s) = (q_0^+ + q_2^- s^2 + \ldots) + (q_1^- + q_3^+ s^3 + \ldots) \qquad (3c)$$

$$K_4(s) = P_{eh}(s) + P_{oh}(s) = (q_0^+ + q_2^- s^2 + \ldots) + (q_1^+ s + q_3^- s^3 + \ldots) \qquad (3d)$$

where P_{el}, P_{eh} (P_{ol}, P_{oh}) are responding to smallest and largest even (odd) part. For clearly understanding the Kharitonov's theorem, we search its geometrical interpretation in frequency-domian through substituting $s=j\varpi$ in (2). For a fixed frequency $\varpi=\varpi_0$, the image of (2) in complex plane is in shape of rectangle (Kharitonov's rectangle) as illustrated in Fig.1 (*a*) and its four vertices are corresponding to four Kharitonov's polynomials. We denote $|P(j\varpi)|$ the distance from the origin to a point within the Kharitonov's rectangle, while $arg\ P(j\varpi)$ stands for the angle between the positive *Re*-axis and the line from the origin to a

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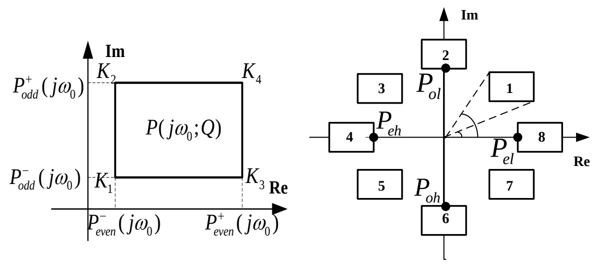
point within the Kharitonov's rectangle. Notice that as ϖ varies, so does the location of the Kharitonov's rectangle. As illustrated in Fig.1(b), cases to get the extreme values of $|P(j\varpi)|$ and $arg\ P(j\varpi)$ fall into 8 different possibilities (exclude the case that



(a) (b)

Fig.1 (a) Kharitonov's rectangle

(b) 8 possibilities of Kharitonov's rectangle

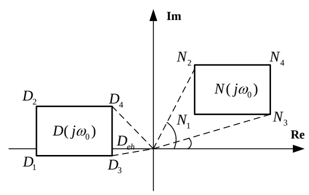
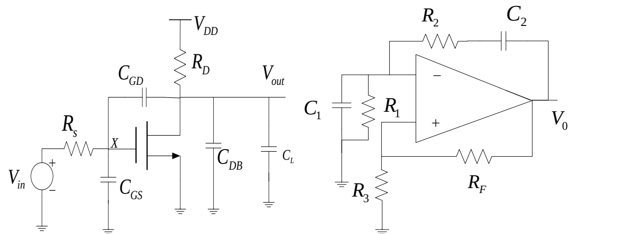


Fig.2 Kharitonov's rectangles for numerator and denominator of interval transfer function



(a) (b)

Fig. 3 (a) Common source amplifier (b) Wien Bridge oscillator

Table1 Parameter and variation range of common source amplifier

parameter	Nominal value	Tolerance range ($\pm 30\%$)
R_D	$10k\Omega$	$[7, 13]\ k\Omega$
C_{DB}	$17.9731fF$	$[12.58, 23.37]\ fF$
g_m	$0.3052mA/V$	$[0.2136, 0.3968]\ mA/V$
C_{GS}	$27.345fF$	$[19.14, 35.55]\ fF$
R_s	$5\ k\Omega$	$[3.5, 6.5]\ k\Omega$
C_L	$1\ \mu F$	$[0.7, 1.3]\ \mu F$
r_0	$390.11\ k\Omega$	$[273.07,507.14]\ k\Omega$
C_{GD}	$1.3897\ fF$	$[0.972, 1.807]\ fF$

Table 2 Parameter and variation range of Wien Bridge oscillator

parameter	Nominal value	Tolerance range ($\pm 30\%$)
R_F	$20k\Omega$	$[14, 26]\ k\Omega$
R_1, R_2, R_S	$10\ k\Omega$	$[7, 13]\ k\Omega$
C_1, C_2	$1nF$	$[0.7,1.3]\ nF$

$P(j\varpi)=0$) . Careful examination of the Kharitonov's rectangle reveals several important facts:

1) $max\ |P(j\varpi)|$, $min\ arg\ P(j\varpi)$ and $max\ arg\ P(j\varpi)$ can be obtained at one of four vertices.
2) $min\ |P(j\varpi)|$ may coincide with the one of { $|K_1(j\varpi)|$, $|K_2(j\varpi)|$, $|K_3(j\varpi)|$, $|K_4(j\varpi)|$, $|P_{el}(j\varpi)|$, $|P_{eh}(j\varpi)|$, $|P_{ol}(j\varpi)|$, $|P_{oh}(j\varpi)|$}.

For detailed discussion, one can refer to [6].

III. Construction of Critical Transfer Function

We consider (1) as the quotient of two families of interval polynomials of (2). Thus, to obtain $|H(j\varpi)|$ and $arg\ H(j\varpi)$, we only need to divide the gains and subtract the phases of the numerator and denominator. For finding the minimum and maximum values, 4 computational formulas are given:

$$max|H(j\omega)| = \frac{\max|N(j\omega)|}{\min|D(j\omega)|} \tag{4a}$$

$$min|H(j\omega)| = \frac{\min|N(j\omega)|}{\max|D(j\omega)|} \tag{4b}$$

$$max\ arg\ H(j\varpi) = max\ arg\ N(j\varpi) - min\ arg\ D(j\varpi) \tag{4c}$$

$$min\ arg\ H(j\varpi) = min\ arg\ N(j\varpi) - max\ arg\ D(j\varpi) \tag{4d}$$

At a given frequency point ϖ_0, two Kharitonov's rectangle for $N(j\varpi_0)$ and $D(j\varpi_0)$ can be sketched as shown in Fig.2. D_1, D_2, D_3, D_4 and N_1, N_2, N_3, N_4 denote the four Kharitonov's polynomials for denominator and numerator, while D_{el}, D_{eh}, D_{ol}, D_{oh} and N_{el}, N_{eh}, N_{ol}, N_{oh} represent smallest even part, largest even part, smallest odd part and largest odd part for denominator and numerator respectively. In this case, $|N(j\varpi_0)|$ is maximized at $|N_4(j\varpi_0)|$ and minimized at $|N_1(j\varpi_0)|$, whereas $max\ arg\ N(j\varpi_0)$ is got at $arg\ N_2(j\varpi_0)$ and $min\ arg\ N(j\varpi_0)$ is equal to $arg\ N_3(j\varpi_0)$. Similarly, $|D(j\varpi_0)|$ is maximized at $|D_2(j\varpi_0)|$ and minimized at $|D_{eh}(j\varpi_0)|$, while $max\ arg\ D(j\varpi_0)$ is got at $arg\ D_3(j\varpi_0)$ and $min\ arg\ D(j\varpi_0)$ is at $arg\ D_4(j\varpi_0)$. Now, they can be combined to calculate the extreme values of $|H(j\varpi_0)|$ and $arg\ H(j\varpi_0)$. That is, $max\ |H(j\varpi_0)|$ is equal to $|N_4(j\varpi_0)|$ / $|D_{eh}(j\varpi_0)|$, whereas $max\ arg\ H(j\varpi_0)$ is the difference between $arg\ N_2(j\varpi_0)$ and $arg\ D_3(j\varpi_0)$; $min\ |H(j\varpi_0)|$ is equal to $|N_1(j\varpi_0)|$ / $|D_2(j\varpi_0)|$, whereas $min\ arg\ H(j\varpi_0)$ is the difference between $arg\ N_3(j\varpi_0)$ and $arg\ D_4(j\varpi_0)$. For a range of frequencies, we should take into consideration the overall possible combinations of the extreme values of numerator and denominator. By taking into consideration all the situations, the following two corollaries can be obtained.

Corollary3.1 The envelopes of 48 critical Kharitonov-type interval transfer functions:

$$H_i(s) = \frac{N_j(s)}{D_k(s)} \tag{5}$$

i=1,2,3,…48;

$N_j \in \{N_1, N_2, N_3, N_4, N_{el}, N_{eh}, N_{ol}, N_{oh}\}$;

$D_k \in \{D_1, D_2, D_3, D_4, D_{el}, D_{eh}, D_{ol}, D_{oh}\}$;

yields variation range of magnitude response .

Corollary3.2 The envelopes of 16 critical Kharitonov-type interval transfer functions:

$$H_i(s) = \frac{N_j(s)}{D_k(s)} \tag{6}$$

i=1,2,3,…16;

$N_j \in \{N_1, N_2, N_3, N_4\}$;

$D_k \in \{D_1, D_2, D_3, D_4\}$;

yields variation range of phase response .

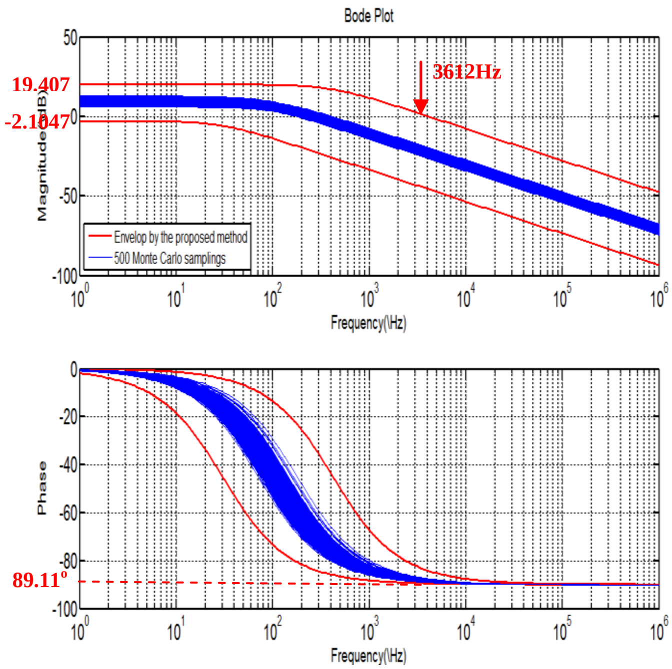


Fig. 4 Simulation result of common source amplifier (envelopes by the proposed method and 500 Monte Carlo samplings)

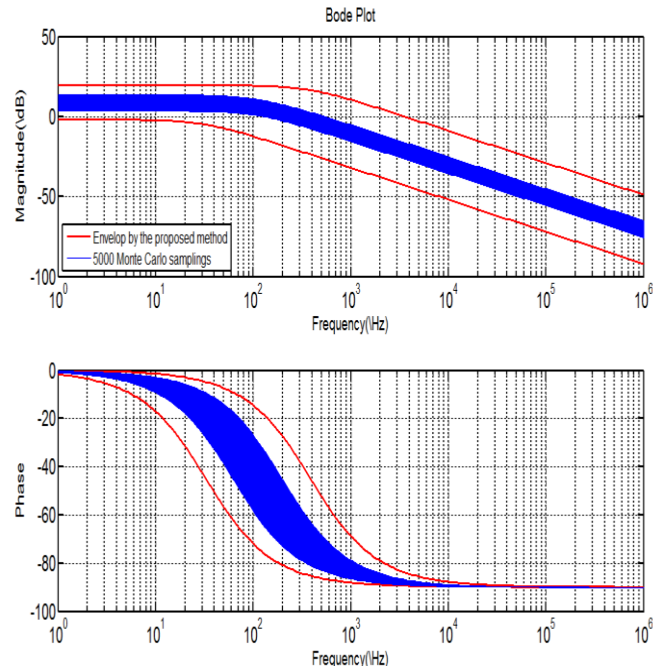


Fig.5 Simulation result of common source amplifier (envelopes by the proposed method and 5000 Monte Carlo samplings)

IV. CASE STUDIES

To illustrate the basic idea of our approach, two cases are studied. One is a common source amplifier, another one is Wien Bridge oscillator.

Case 1. Consider a common source amplifier of Fig.3 (*a*), its transfer function of output/input voltage is given:

$$\frac{V_{out}}{V_{in}}(s) = \frac{(C_{GD}s - g_m)R_D^{'}}{R_S R_D^{'}\xi s^2 + [\zeta + R_s C_{GS} + R_D^{'}(C_{GD} + C_{DB} + C_L)]s + 1} \quad (7)$$

Where $\xi = C_{GS}C_{GD} + C_{GS}(C_{DB}+C_L) + (C_{DB}+C_L)C_{GD}$, $R_D^{'} = R_D // r_0$, $\zeta = R_S C_{GD}(1 + g_m R_D^{'})$. Assume that there is ±30% variation to every parameter. The nominal value and tolerance range of each electrical parameter is listed in Table. 1. The variation ranges of parameters are firstly mapped into coefficients' variation ranges by applying interval arithmetic. Given two real interval numbers p_1 and p_2, then the basic interval arithmetic operations are:

1) $p_1 + p_2 = [p_1^- + p_2^-, p_1^+ + p_2^+]$;
2) $p_1 - p_2 = [p_1^- - p_2^+, p_1^+ - p_2^-]$;
3) $p_1 \times p_2 = [min(p_1^- p_2^-, p_1^- p_2^+, p_1^+ p_2^-, p_1^+ p_2^+), max(p_1^- p_2^-, p_1^- p_2^+, p_1^+ p_2^-, p_1^+ p_2^+)]$;
4) $p_1 / p_2 = p_1 \times (1/p_2); p_2 \neq 0$

For example, the first-order coefficient of numerator of (7) is $C_{GD} \times R_D^{'}$, the corresponding variation range of a_1 is the product of C_{GD} and R'_D, that is $[0.0357\times10^{10}, 0.4253\times10^{-10}]$. The remaining coefficients' variation ranges are $[-9.3401, -0.7851]$ for a_0, $[0.0025, 0.0307]$ for b_1, and $[0.0175\times10^{-12}\ 0.7187\times10^{-12}]$ for b_2, respectively. Secondly, the eight Kharitonov's polynomials for numerator and denominator are calculated in equation (8*a*) to (8*h*):

$$N_1(s) = N_{el}(s) + N_{ol}(s) = -9.3401 + 0.0357\times10^{-10}s \quad (8a)$$
$$N_2(s) = N_{el}(s) + N_{oh}(s) = -9.3401 + 0.4253\times10^{-10}s \quad (8b)$$
$$N_3(s) = N_{eh}(s) + N_{ol}(s) = -0.7851 + 0.0357\times10^{-10}s \quad (8c)$$
$$N_4(s) = N_{eh}(s) + N_{oh}(s) = -0.7851 + 0.4253\times10^{-10}s \quad (8d)$$
$$D_1(s) = D_{el}(s) + D_{ol}(s) = 1 + 0.7187\times10^{-12}s^2 + 0.0025s \quad (8e)$$
$$D_2(s) = D_{el}(s) + D_{oh}(s) = 1 + 0.7187\times10^{-12}s^2 + 0.0307s \quad (8f)$$
$$D_3(s) = D_{eh}(s) + D_{ol}(s) = 1 + 0.0175\times10^{-12}s^2 + 0.0025s \quad (8g)$$
$$D_4(s) = D_{eh}(s) + D_{oh}(s) = 1 + 0.0175\times10^{-12}s^2 + 0.0307s \quad (8h)$$

Following corollary 3.1 and corollary 3.2, 48 and 16 critical Kharitonov-type transfer functions are readily constructed from (8*a*) to (8*h*). The calculated worst case variation ranges of magnitude and phase response are plotted in red lines in Fig.4 and Fig.5. We compare the envelopes with 500 Monte Carlo simulation samples in Fig.4 and 5000 Monte Carlo samplings in Fig.5, respectively. Their parameters are randomly sampled from corresponding variation ranges shown in Table.1. It is clearly illustrated that the Monte Carlo samplings are well enclosed by the envelopes. The more Monte Carlo samplings are been sampled, the closer they are to the envelopes, and they will finally hit the envelopes if sufficient samplings are simulated. From the calculated envelopes, we can tell that because of the process variations, the gain of the common source amplifier may be as high as 19.407dB, the highest UGF may be 3612Hz, and the highest phase margin may be 89.1°. The envelopes also predict that the common source amplifier may lose the ability of amplification, since the lowest gain may be -2.1047dB. It provides an useful information to designers to optimize their circuit by considering likely performance under process variations.

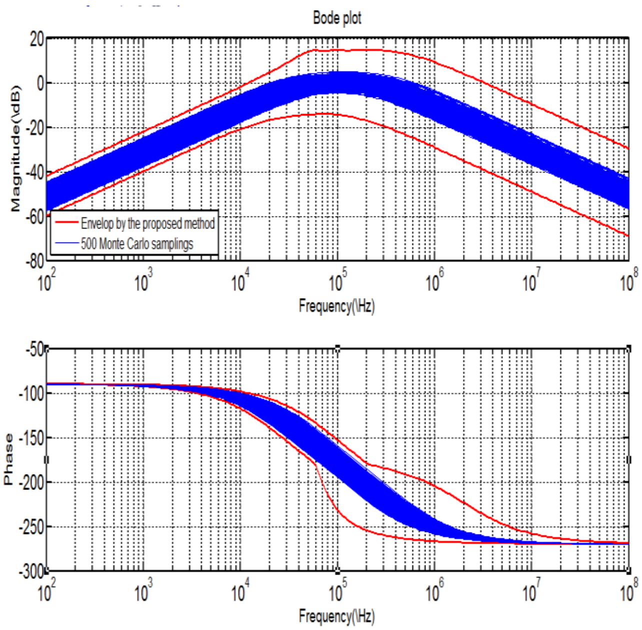


Fig.6 Simulation result of Wien Bridge oscillator (envelopes by the proposed method and 500 Monte Carlo samplings)

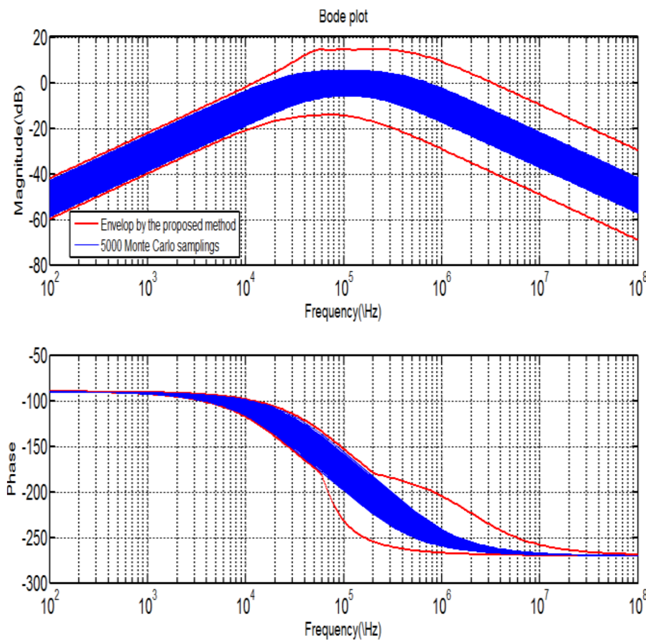


Fig.7 Simulation result of Wien Bridge oscillator (envelopes by the proposed method and 5000 Monte Carlo samplings)

Case 2. Consider a Wien Bridge oscillator shown in Fig.3 (*b*). The transfer function is given as:

$$\frac{V_0}{V'_0}(s) = \frac{(1+\frac{R_F}{R_S})R_1C_2s}{R_1R_2C_1C_2s^2+(R_1C_1+R_2C_2+R_1C_2)s+1} \quad (9)$$

With assumption of ±30% variation level to every parameter, Table.2 lists the nominal value and variation range of every parameter. A similar transformation from variation ranges of parameters to the coefficients' variation ranges is applied at first, and they are $[1.0177\times10^{-5}, 7.9672\times10^{-5}]$ for a_1, $[1.47\times10^{-5}, 5.07\times10^{-5}]$ for b_1, $[2.401\times10^{-11}, 2.8561\times10^{-10}]$ for b_2. Next, the eight Kharitonov's polynomials for numerator and numerator can be formulated as:

$$N_{ol}(s) = 1.0177\times10^{-5}s \quad (10a)$$
$$N_{oh}(s) = 7.9672\times10^{-5}s \quad (10b)$$
$$D_1(s) = D_{el}(s)+D_{ol}(s)=1+2.8561\times10^{-10}s^2+1.47\times10^{-5}s \quad (10c)$$
$$D_2(s) = D_{el}(s)+D_{oh}(s)=1+2.8561\times10^{-10}s^2+5.07\times10^{-5}s \quad (10d)$$
$$D_3(s) = D_{eh}(s)+D_{ol}(s)=1+2.401\times10^{-11}s^2+1.47\times10^{-5}s \quad (10e)$$
$$D_4(s) = D_{eh}(s)+D_{oh}(s)=1+2.401\times10^{-11}s^2+5.07\times10^{-5}s \quad (10f)$$

Since there is only one item in the numerator of (9), the numbers of critical Kharitonov-type transfer functions are reduced to 20 for magnitude response and 8 for phase response. Fig.6 and Fig.7 display the calculated worst case performance ranges of magnitude and phase responses of Wien Bridge oscillator in red lines, and 500 Monte Carlo samplings and 5000 Monte Carlo samplings in blue lines. The more samplings are simulated, the closer they are to the envelopes. The envelopes predict that the frequency which may cause oscillation varies from 1.8834 *Hz* to 6.4961 *Hz*, when the Monte Carlo samples fail to evaluate the likely performance range for Wien bridge oscillator within finite samplings.

V. Conclusion

Two practical VLSI circuits are studied to demonstrate the efficiency of the proposed approach in performance robustness analysis of VLSI circuits with process variations. This method takes advantage of Kharitonov's theorem and reduces the computational complexity to at most 48 Kharitonov-type critical transfer functions' calculation even for the complicated VLSI circuits with large transfer function order. The approach provides a good graphic means of investigating of worst case frequency-domain performance of VLSI circuits with manufactured variations. The proposed method is general and can be extended to evaluate worst case performance for other circuits and other performance metrics of interest when a similar interval transfer function is established.

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