An Analysis of Curvatures of Discrete Surfaces with Boundary

Honors Undergraduate Thesis*

UMBC

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Abstract

While there is a rich literature on methods for computing the curvature of discrete surfaces, the case of discrete surfaces with boundary is often ignored. We show how to compute curvature measures for discrete surfaces with boundary using the theory of normal cycles. Our curvature formulae for discrete surfaces exhibit boundary terms not seen in previous formulae. We also derive error bounds comparing the curvature measure of a discrete surface to the curvature measure of a smooth surface that it approximates.

1 Introduction

Understanding the curvature of anatomical surfaces is important for biomedical applications. Examples of anatomical surfaces include heart and brain surfaces. Quantifying the structure of these anatomical surfaces can enhance our understanding of anatomical function and help us identify diseases. Anatomical surfaces can be studied by extracting discrete surfaces approximating the anatomical surface from 3D imaging data. We represent these discrete surfaces using triangular surface meshes, i.e. a surface formed as the union of triangles whose vertices lie on our discrete surface. Thus the problem of computing the curvature of anatomical surfaces reduces to the question of how to compute the curvature of discrete surfaces.

There is a rich literature of methods for estimating the curvature of smooth surfaces using approximating discrete surfaces. One approach is to use finite element analysis, approximating the curvature at a vertex using approximating polynomials [14]. An approach combining finite element and finite volume analysis has been used by Meyer, at el., to approximate the

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Beltrami-Laplace operator and curvature [7]. Another common approach is to compute curvature measures which quantify the average curvature over regions in a surface [13], [10]. We generalize an approach due to Cohen-Steiner and Morvan that uses the concept of a normal cycle from geometric measure theory to compute curvature measures [10]. The advantage of this approach is that the curvature of a large class of surfaces, including both smooth and discrete surfaces, can be computed in terms of normal cycles. Another advantage is that Cohen-Steiner and Morvan showed how to use geometric measure theory to compute error bounds on the difference of the curvature measures of a smooth surface and an approximating discrete surface. However, Cohen-Steiner and Morvan developed their theory only for smooth and discrete surfaces without boundary. When studying anatomical surfaces, one often considers surfaces with boundary in order, for example, to study localized regions of the brain and to avoid the problem of large-scale heterogeneties in the data due to imperfections in the MRI data collection process. Thus, it is important to have a principled approach to the computation of the curvature of discrete surfaces with boundary, complete with error bounds on curvature estimates.

In this thesis, we will show how to compute the curvature of discrete surfaces with boundary using normal cycles. We will focus on computing the Gauss curvature and mean curvature normal of surfaces with boundary. Gauss curvature is an intrinsic property of surfaces which can be computed in terms of distance and angles on the surface and is independent of how the surface is embedded in three-dimensional space. Mean curvature is important in area minimization problems and in smoothing algorithms. For technical reasons, we analyze the mean curvature normal, which is a vector-valued curvature whose length is the mean curvature at each point. We also show how to compute the vector second fundamental form, which is a bilinear form that encodes the principle curvatures of a surface. The thesis concludes with a proof of a convergence theorem describing how the curvature of a discrete surface converges to the curvature of a smooth surface that it approximates as the mesh size converges to zero. This convergence theorem parallels the error bounds given by Cohen-Steiner and Morvan [9]. However, the theorem and its proof are more involved for surfaces with boundary.

The structure of this thesis is as follows. We will begin in Section 1 with an overview of basic concepts in differential geometry and geometric measure theory. In particular, we will discuss how curvature is computed on smooth surfaces. In Section 2, we provide concrete definitions of the normal cycle of smooth and discrete surfaces with boundary. These definitions are consistent with the abstract definition of the normal cycle of subanalytic sets given in [3]. In Sections 3, 4, and 5, we use the normal cycle to compute explicit formulae for the Gauss curvature, mean curvature normal, and vector second fundamental form for smooth and discrete surfaces with boundary. In Section 6, we state and prove the convergence theorem. We will first prove a general convergence theorem comparing the curvature measures of a region on a discrete surface and its image under a Lipschitz homeomorphism from the discrete surface to the smooth surface. We will then derive a more explicit version of the theorem using an explicit homeomorphism between the discrete and smooth surfaces.

2 Background material

2.1 Smooth manifolds in $\mathbb{R}^m$

We begin by considering the concept of a manifold. We formally define an *n-dimensional topological manifold* M to be a topological space such that for each point $p \in M$, there is an open neighborhood $\mathcal{U}$ of p in M and a continuous bijective mapping $\mathbf{x} : D \to \mathcal{U}$, where

D is an open set in $\mathbb{R}^n$. In other words, we can parameterize M locally using maps $\mathbf{x}$. We will only be considering submanifolds of Euclidean spaces $\mathbb{R}^m$. In this case, we say a map $\mathbf{x} : D \to \mathbb{R}^m$, $D \subseteq \mathbb{R}^n$, is *smooth* if $\mathbf{x}$ is infinitely differentiable and we say $\mathbf{x}$ is *regular* if its Jacobian has rank n [12, p. 38]. A topological manifold $M \subseteq \mathbb{R}^m$ is a *smooth manifold* if for every point $p \in M$ there is smooth regular map $\mathbf{x} : D \to M$, where $D \subseteq \mathbb{R}^n$ is open, such that $p \in \mathbf{x}(D)$. The smooth regular maps $\mathbf{x}$ are called *patches* [12, p. 125-126]. Examples of manifolds in $\mathbb{R}^3$ include curves, surfaces, and $\mathbb{R}^3$ itself. We say a mapping $f : M_1 \to M_2$ between smooth manifolds is a smooth map if $\mathbf{y}^{-1} \circ f \circ \mathbf{x}$ is infinitely differentiable for any patches $\mathbf{x}$ of M_1 and $\mathbf{y}$ of M_2 [12, p. 160].

We will be interested in studying surfaces such as the closed unit disk which is not included in our current definition of a smooth manifold. Thus we define an *n-dimensional manifold with boundary* to be a topological space M for which given a point $p \in M$, there is an open neighborhood $\mathcal{U}$ of p in M and a continuous bijective mapping $\mathbf{x} : D \to \mathcal{U}$, where D is either an open set in $\mathbb{R}^n$ or an open set of the half plane $\{(x_1, ..., x_n) : x_n > 0\}$ [11, p. 200]. As before, we say M is a smooth manifold with boundary if we can always choose such maps $\mathbf{x}$ so that they are smooth and regular and we will refer to the smooth regular maps $\mathbf{x}$ as patches [11, p. 201]. We say a point $p \in M$ lies in the interior of M if p there is an open neighborhood $\mathcal{U}$ of p in M and a continuous bijective mapping $\mathbf{x} : D \to \mathcal{U}$, where D is an open set in $\mathbb{R}^n$. The set of all points that lie in the interior of M is called the *interior* of M and is denoted by $\operatorname{int} M$. We say a point $p \in M$ lies on the boundary of M if $p \notin \operatorname{int} M$. The set of all points p that lie on the boundary of M is called the *boundary* of M and is denoted by ∂M [11, p. 205]. If $\partial M = \emptyset$, we say M is a *manifold without boundary.* For every point $p \in \partial M$, there is a map $\mathbf{x} : [0, 1) \times (0, 1)^{n-1} \to M$ such that $p = \mathbf{x}(0, u_1, ..., u_{n-1}) \in \partial M$ for some $u_i \in (0, 1)$, $i = 1, 2, ..., n - 1$. When M has a boundary, ∂M is an $(n - 1)$-dimensional manifold since the map $\mathbf{y} : (-1, 1)^{n-1} \to M$ defined by $\mathbf{y}(u_1, ..., u_{n-1}) = \mathbf{x}(0, u_1, ..., u_{n-1})$ is a patch for ∂M.

To study the geometry of a manifold, we introduce the concept of a vector. A *vector* $\vec{v}$ at a point $p \in \mathbb{R}^m$ can be regarded as a directed line segment from p to the point $p + \vec{v}$. The set of all tangent vectors at p will be denoted by $T_p\mathbb{R}^m$. We know $T_p\mathbb{R}^m$ is a vector space isomorphic to $\mathbb{R}^m$ under the canonical isomorphism mapping the directed line segment from p to the point $p + \vec{v}$ to the point $\vec{v}$ in $\mathbb{R}^m$. Given any n-dimensional manifold M in $\mathbb{R}^m$, we define T_pM to be the set of velocity vectors at p to curves lying in M and passing through p. Note that T_pM is an n-dimensional vector subspace of $T_p\mathbb{R}^m$. Given a patch $\mathbf{x} : D \to M$, $D \subset \mathbb{R}^n$, the set of vectors $\left\{\frac{\partial \mathbf{x}}{\partial x_i} : i = 1, 2, ..., n\right\}$ is a basis for T_pM. The *tangent bundle* to M is the the manifold TM defined by $TM = \bigcup_{p \in M} T_pM$. There is a projection map $\pi : TM \to M$ defined by $\pi(\vec{v}) = p$ for all vectors $\vec{v} \in T_pM$. We define a *vector field* on M to be a mapping $\vec{X} : M \to TM$ that assigns to every point $p \in M$ a vector $\vec{X}(p) \in T_pM$. A vector field $\vec{X}$ is a smooth if it is a smooth mapping between manifolds $\vec{X} : M \to TM$. We can also regard vectors and vector fields to be differential operators on smooth functions $f : M \to \mathbb{R}$. Given a vector field $\vec{X}$ on $M \subseteq \mathbb{R}^m$, $\vec{X}(f) : M \to \mathbb{R}$ is the function whose value at a point $p \in M$ is the directional derivative of f in the direction $\vec{X}(p)$.

Let $M \subseteq \mathbb{R}^m$ be an n-dimensional smooth manifold. A *frame* at $p \in M$ is an ordered n-tuple of n linearly independent vectors $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$, where $\vec{v}_j \in T_pM$ for all j. An *orientation* at $p \in M$ is an equivalence class of frames which is defined as follows. Given two frames $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ and $(\vec{w}_1, \vec{w}_2, ..., \vec{w}_n)$, let P be the change of basis matrix from the basis $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ to the basis $\{\vec{w}_1, \vec{w}_2, ..., \vec{w}_n\}$. We say $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ and $(\vec{w}_1, \vec{w}_2, ..., \vec{w}_n)$ have the same orientation if $\det(P) > 0$ and $(\vec{v}_1, \vec{v}_2, ..., \vec{v}_n)$ and $(\vec{w}_1, \vec{w}_2, ..., \vec{w}_n)$ have the different orientations if $\det(P) < 0$. Thus orientation is an equivalence relation on frames with precisely two equivalence classes. We often choose one equivalence class so that all the frames in that equivalence class are said to be *positively oriented* and all the frames in the other equivalence class are said to be *negatively oriented*. We say that $M \subseteq \mathbb{R}^m$ is *orientable* if there is choice of orientation at each point of M such that given any two continuous frame fields $(\vec{X}_1, \vec{X}_2, ..., \vec{X}_n)$ and $(\vec{Y}_1, \vec{Y}_2, ..., \vec{Y}_n)$ on a connected open subset $\mathcal{U} \subseteq M$, either $(\vec{X}_1(p), \vec{X}_2(p), ..., \vec{X}_n(p))$ and $(\vec{Y}_1(p), \vec{Y}_2(p), ..., \vec{Y}_n(p))$ have the same orientation at every $p \in \mathcal{U}$ or $(\vec{X}_1(p), \vec{X}_2(p), ..., \vec{X}_n(p))$ and $(\vec{Y}_1(p), \vec{Y}_2(p), ..., \vec{Y}_n(p))$ have different orientations at every $p \in \mathcal{U}$. For the rest of this paper we will only consider smooth manifolds that are orientable. In the case that M is an oriented surface in $\mathbb{R}^3$, we can associate an orientation with a choice of a unit normal field $\vec{U}$ on M by declaring a frame $(\vec{X}_1, \vec{X}_2)$ at $p \in M$ positively oriented if $\vec{U}(p)$ and $\vec{X}_1 \times \vec{X}_2$ have the same direction. The boundary of an oriented manifold M is also an oriented manifold. Suppose $M \subseteq \mathbb{R}^3$ is a orientable smooth surface with boundary and let $(\vec{T}, \vec{V}, \vec{U})$ be the orthonormal frame field, called the Darboux frame, on ∂M such that $\vec{T}$ is tangent to ∂M, $\vec{V}$ is tangent to M and directed into M, and $\vec{U}$ is the positive unit normal field to M. We can define an orientation on ∂M by letting $\vec{T}$ be a positively oriented vector field on ∂M if $(\vec{T}, \vec{V})$ is a positively oriented frame field on M.

2.2 Differential forms

Before discussing differential forms, we will consider the general concept of a dual vector space. Given a vector space V, we define the *dual vector* space of V to be $V^* = \mathrm{Hom}(V, \mathbb{R})$, the set of all linear transformations from V to $\mathbb{R}$. Given a basis $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ of V, we define the dual basis $\{\omega_1, \omega_2, ..., \omega_n\}$ for V^* so that $\omega_i(\vec{v}_j) = 1$ if $i = j$ and $\omega_i(\vec{v}_j) = 0$ if $i \neq j$. It can be shown that a dual basis is in fact a basis for V^* and as a consequence V^* is isomorphic to V.

Now let M be an n-dimensional manifold in $\mathbb{R}^m$ and p be a point in M. The space of *1-forms* on M at p, denoted by T_p^*M, is the dual vector space of T_pM. An element $\omega_p \in T_p^*M$ is called a 1-form and is a linear functional on T_pM which assigns to each vector $\vec{v} \in T_pM$ a real number $\omega_p(\vec{v})$. A common example of a 1-form on $\mathbb{R}^3$ is the differential dg of a smooth function $g : \mathbb{R}^3 \to \mathbb{R}$. We define dg by $dg(\vec{v}) = \vec{v}(g)$ for all $v \in T_p\mathbb{R}^3$. Since $(\lambda\vec{v} + \vec{w})(g) = \lambda\vec{v}(g) + \vec{w}(g)$ for all vectors $\vec{v}, \vec{w} \in T_p\mathbb{R}^3$ and for any scalar $\lambda \in \mathbb{R}$, dg is a linear operator on vectors. We can consider the differentials of the coordinate functions x, y, and z. Given a vector $\vec{v} = (v_1, v_2, v_3)$, the 1-forms dx, dy, dz satisfy $dx(\vec{v}) = v_1$, $dy(\vec{v}) = v_2$, and $dz(\vec{v}) = v_3$. One can show that at a point $p \in \mathbb{R}^3$, dx, dy, dz form a dual basis for $T_p^*\mathbb{R}^3$ corresponding to the basis $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ of $T_p\mathbb{R}^3$. Hence we can represent every 1-form $\omega_p \in T_p^*\mathbb{R}^3$ in the familiar form $\omega_p = g_1dx + g_2dy + g_3dz$, where the g_i are real-valued functions of the points p in M.

The space $T^*M = \bigcup_{p \in M} T_p^*M$ is the cotangent bundle to M and is a $2n$-dimensional manifold. There is a projection map $\pi : T^*M \to M$ defined by $\pi(\omega) = p$ for all vectors $\omega \in T_p^*M$. To every vector field $\vec{X}$ on M, we can associate a 1-form on M, also denoted by $\vec{X}$, defined by $\vec{X}(\vec{Y}) = \vec{X} \bullet \vec{Y}$ for all $\vec{Y} \in T\mathbb{R}^3$. At each point $p \in M$, the mapping from the vector $\vec{v} \in T_pM$ to the corresponding 1-form $\vec{v} \in T_p^*M$ is an isomorphism between T_pM and T_p^*M. In particular, given an orthonormal basis $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ of T_pM, the set of corresponding 1-forms $\{\vec{v}_1, \vec{v}_2, ..., \vec{v}_n\}$ is the corresponding dual basis for T_p^*M.

An important operation on 1-forms is the pullback operation. Let $f : M_1 \to M_2$ be a smooth map between smooth manifolds M_1 and M_2. For a vector $\vec{v} \in T_pM_1$, we define the *pushforward* of $\vec{v}$ to be the vector $f_*\vec{v} \in T_pM_2$ such that for any smooth function $g : M_2 \to \mathbb{R}$, $f_*\vec{v}(g) = \vec{v}(g \circ f)$. By duality, we define the *pullback* $(f^*\omega)_p$ of a 1-form $\omega_{f(p)} \in T^*_{f(p)}M_2$ to be the 1-form in $T_p^*M_1$ such that $(f^*\omega)_p(\vec{v}) = \omega_{f(p)}(f_*\vec{v})$ for any vector $\vec{v} \in T_pM_1$.

We will primarily be working with 1-forms of the following form on $T\mathbb{R}^3$. We will regard $T\mathbb{R}^3$ as $T\mathbb{R}^3 \simeq E_p \times E_n$, where $E_p \simeq \mathbb{R}^3$ is called the *point space* and $E_n \simeq \mathbb{R}^3$ is called the *normal space*. An element in $T\mathbb{R}^3$ can therefore be written as (p, n), where $p \in E_n$ is a point in space and $n \in E_n$ is a vector at the point p. We will also study the manifold $T\mathbb{R}^3$. A vector in $T_{(p,n)}(T\mathbb{R}^3)$ can be written as $(\vec{v}_1, \vec{v}_2)$, where $\vec{v}_1 \in TE_p$ and $\vec{v}_2 \in TE_n$. Since $T\mathbb{R}^3 \simeq E_p \times E_n$, the differential form associated with $(\vec{v}_1, \vec{v}_2) \in T(T\mathbb{R}^3)$, which we will also denote by $(\vec{v}_1, \vec{v}_2)$ satisfies $(\vec{v}_1, \vec{v}_2)((\vec{w}_1, \vec{w}_2)) = \vec{v}_1 \bullet \vec{w}_1 + \vec{v}_2 \bullet \vec{w}_2$ for all $(\vec{w}_1, \vec{w}_2) \in T(T\mathbb{R}^3)$.

A *2-form* ω is a bilinear functional which at every point $p \in M$ assigns to every pair of vectors $\vec{v}_1, \vec{v}_2 \in T_pM$ a real number $\omega_p(\vec{v}_1, \vec{v}_2)$ such that $\omega_p(\vec{v}_1, \vec{v}_2) = -\omega_p(\vec{v}_2, \vec{v}_1)$. Given 1-forms ω_1 and ω_2, we define the 2-form $\omega_1 \wedge \omega_2$ by

$$(\omega_1 \wedge \omega_2)(\vec{v}_1, \vec{v}_2) = \begin{vmatrix} \omega_1(\vec{v}_1) & \omega_2(\vec{v}_1) \\ \omega_1(\vec{v}_2) & \omega_2(\vec{v}_2) \end{vmatrix}.$$

Given a smooth map $f : M_1 \to M_2$ between smooth manifolds M_1 and M_2 and a 2-form ω on M_2, we define the pullback of ω to be the 2-form $f^*\omega$ defined by $(f^*\omega)(v_1, v_2) = \omega(f_*\vec{v}_1, f_*\vec{v}_2)$. We define the exterior differentive operator d from k-forms to $(k+1)$-forms to be the unique operator that satisfies the following properties:

1. $d(a_1\omega_1 + a_2\omega_2) = a_1 d\omega_1 + a_2 d\omega_2$ for k-forms ω_1 and ω_2 and $a_1, a_2 \in \mathbb{R}$,
2. $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^j \omega_1 \wedge d\omega_2$ for any j-form ω_1 and k-form ω_2, and
3. $d(d\omega) = 0$ for any differential form ω.

Thus the exterior derivative a 1-form $\omega = \sum_{i=1}^m g_i dx_i$ on $\mathbb{R}^m$ is the 2-form $d\omega = \sum_{i=1}^m dg_i \wedge dx_i$. Given a smooth map $f : M_1 \to M_2$ between smooth manifolds M_1 and M_2, the chain rule implies $d(f^*\omega) = f^*d(\omega)$ for any k-form ω.

2.3 Triangulated surfaces in $\mathbb{R}^3$

A *triangulated surface* $T \subseteq \mathbb{R}^3$ is a 2-dimensional topological manifold that is the finite union of closed triangles. The closed triangles will be referred to as the *faces* of T. We further require that any two distinct faces of T must either be disjoint, intersect each other at a common edge, or intersect each other at a common vertex. The closure of the edges of the faces of T are called the *edges* of T and the vertices of the faces of T are called the *vertices* of T.

An *orientation* of a triangular face of T is an ordering v_1, v_2, v_3 of the vertices of the triangle. We denote such an *oriented triangle* by $[v_1, v_2, v_3]$. We say two orientations are the same if they differ from each other by an even permutation of the vertices. For example, the oriented triangles $[v_1, v_2, v_3]$ and $[v_2, v_3, v_1]$ have the same orientation, where as $[v_1, v_2, v_3]$ and $[v_1, v_3, v_2]$ have opposite orientations. An orientation of an edge of a triangle is an ordering v_1, v_2 of the vertices of the edge. We will denote such an *oriented edge* by $[v_1, v_2]$. An edge with vertices v_1 and v_2 has precisely two orientations, $[v_1, v_2]$ and $[v_2, v_1]$, so we say the oriented edges $[v_1, v_2]$ and $[v_2, v_1]$ have opposite orientations. An oriented triangle induces an orientation on its edges by $[v_1, v_2]$, $[v_2, v_3]$, and $[v_3, v_1]$ being the oriented edges of $[v_1, v_2, v_3]$. We say a triangulated surface T is *orientable* if each face of T can be oriented so that whenever two faces share a common a edge, the orientations induced by the faces on the edge are opposite orientations.

2.4 Currents

A *k-dimensional current* S on M is a linear functional which assigns to every smooth k-form ω on M a real number $S(\omega)$. We will often write $\langle S, \omega \rangle = S(\omega)$. The *degree* of a k-dimensional current on $\mathbb{R}^m$ is $m - k$. The *support*, $\operatorname{spt} S$, of a current S is the smallest closed set such that if a differential form ω is zero almost everywhere on $\operatorname{spt} S$, then $S(\omega) = 0$. We restrict our attention to k-dimensional currents with compact support to ensure that $S(\omega)$ is finite for all smooth k-forms ω.

Given two k-dimensional currents S_1 and S_2 and scalars $\alpha_1, \alpha_2 \in \mathbb{R}$, $\alpha_1 S_1 + \alpha_2 S_2$ is the current defined by $(\alpha_1 S_1 + \alpha_2 S_2)(\omega) = \alpha_1 S_1(\omega) + \alpha_2 S_2(\omega)$. Given a map $f : M_1 \to M_2$ between smooth manifolds M_1 and M_2, the *pushforward* of a current S on M_1 is the current $f_\# S$ on M_2 defined by $f_\# S(\omega) = S(f^* \omega)$. We define the *boundary* of a k-current S by $\partial S(\omega) = S(d\omega)$. A *cycle* is a current S such that $\partial S = 0$. Given a k-dimensional current S and a smooth j-form η with $j \leq k$, $(\eta \wedge S)$ is the $(k-j)$-dimensional current defined by $(\eta \wedge S)(\omega) = (-1)^{jk} S(\eta \wedge \omega)$.

We can associate to every closed, compact, oriented smooth k-dimensional manifold M in $\mathbb{R}^m$ a current $[M]$ defined by

$$[M](\omega) = \int_M \omega,$$

for any k-form ω. Also, for a point $p \in \mathbb{R}^m$, we define $[p]$ the be the 0-dimensional current such that $[p](g) = g(p)$ for any function $f : M \to \mathbb{R}$. In the context of currents associated with a surface $M \subset \mathbb{R}^3$, the operations defined above have special interpretations. Given an injective map $f : M \to \mathbb{R}^3$, the pushforward of $[M]$ satisfies $f_\#[M] = [f(M)]$ and thus for any 2-form ω,

$$[f(M)](\omega) = f_\#[M](\omega) = \int_M f^* \omega.$$

Also, if $\mathbf{x} : [0,1]^2 \to M$ is a coordinate chart on M, then for any 2-form ω,

$$[M](\omega) = \int_M \omega = \int_{[0,1]^2} \mathbf{x}^* \omega = \int_0^1 \int_0^1 \omega(\mathbf{x}_x, \mathbf{x}_y) dx dy,$$

where $\mathbf{x}_x$ and $\mathbf{x}_y$ denote the partial derivatives of $\mathbf{x}$ with respect to x and y, respectively. By Stokes' Theorem, the boundary of $[M]$ and the current $[\partial M]$ that is associated with the boundary of M are equal since

$$\partial[M](\omega) = \int_M d\omega = \int_{\partial M} \omega = [\partial M](\omega)$$

for any $(n-1)$-form ω. Given a differential form ω and smooth manifold of the same dimension, $\omega \wedge [M]$ is the zero dimensional current such that

$$(\omega \wedge [M])(f) = \int_M f \cdot \omega$$

for any function $f : M \to \mathbb{R}$.

There are many special spaces of currents, among them the space of rectifiable currents. The general definition of a rectifiable current is quite technical. However, for our purposes it suffices to know that if S is a k-dimensional current of the form

$$S = \sum_{i=1}^{t} a_i [M_i], \tag{1}$$

where the M_i are k-dimensional closed, compact, oriented manifolds (possibly with boundary) and each a_i is an integer, then S is a *rectifiable* current. A current S is an *integral* current if S and ∂S are both rectifiable currents. Hence, any current S of the form given by (1) is also an integral current. A current S of the form given by (1) is also an example of a current that is *representable by integration*. Given a Borel set $B \subseteq \mathbb{R}^m$ and a current S that is representable by integration, we define the current S *restricted to* B, denoted by $S \llcorner B$, by

$$(S \llcorner B)(\omega) = S(\chi_B \cdot \omega),$$

where $\chi_B : \mathbb{R}^M \to \{0, 1\}$ is the characteristic function for B [2, 4.1.7]. In the case that S of the form given by (1), $S \llcorner B$ is given by

$$(S \llcorner B)(\omega) = \sum_{i=1}^{t} a_i \int_{M_i \cap B} \omega.$$

2.5 Curvature

All information about the curvature of a smooth surface M in $\mathbb{R}^3$ is encoded in the *second fundamental form*. The scalar second fundamental h_p of M at a point p is the symmetric bilinear form on tangent vectors satisfying $h_p(\vec{X}, \vec{Y}) = -\nabla_X \vec{U} \cdot \vec{Y}$ for all $X, Y \in T_pM$, where $\vec{U}$ is the positively oriented unit normal vector field on M [5, p. 139]. The eigenvalues κ_1 and κ_2 of the linear operator associated with h_p are called the *principal curvatures* and the corresponding eigenvectors $\vec{e}_1$ and $\vec{e}_2$ are called the *principal directions*. By the Spectral Theorem, may write the second fundamental form as

$$h = \kappa_1 \vec{e}_1 \otimes \vec{e}_1 + \kappa_2 \vec{e}_2 \otimes \vec{e}_2,$$

where for vectors $\vec{E}, \vec{X}, \vec{Y} \in T_pM$, we define $(\vec{E} \otimes \vec{E})(\vec{X}, \vec{Y}) = \vec{E}(\vec{X})\vec{E}(\vec{Y}) = (\vec{E} \bullet \vec{X})(\vec{E} \bullet \vec{Y})$.

From this information, we can determine the Gauss and mean curvatures of a smooth surface M. The *Gauss curvature* $K(p)$ at a point $p \in M$ is the determinant of the second fundamental form and is equal to $\kappa_1 \kappa_2$. The *mean curvature* $H(p)$ is one half the trace of the second fundamental form and is equal to $\frac{1}{2}(\kappa_1 + \kappa_2)$.

It will also be helpful to describe the curvature of an oriented curve in a smooth surface or on its boundary. Let $\gamma \subset M$ be an oriented smooth curve. The *Darboux frame field* $(\vec{T}, \vec{V}, \vec{U})$ along γ is the positively oriented orthonormal frame field along γ such that $\vec{T}$ is the positively unit tangent vector field to γ and $\vec{U}$ is the positively oriented unit normal vector field to M along γ. We define the geodesic curvature κ_g of γ to be equal to $D_{\vec{T}} \vec{T} \bullet \vec{V}$ [12, p. 337]. One can think of the geodesic curvature of γ as a measure of how much γ is bending inside the surface M.

3 Normal cycles

To study the curvature of a triangulated surface, we use the concept of the normal cycle of the surface. The normal cycle of a smooth or triangulated surface $M \subset \mathbb{R}^3$ is an integral current on $T\mathbb{R}^3$ and is denoted by $N(M)$. Intuitively, we can think of $N(M)$ as the integral current associated with the set of unit normal vectors to M; however, the normal vector at a point where a surface is not smooth is not defined. An abstract definition of $N(M)$ is given by Joe Fu for the large class of subanalytic subsets M of $\mathbb{R}^n$ [3]. In particular, Joe Fu's definition of a normal cycle applies to triangulated surfaces, which are all subanalytic sets. The normal cycle satisfies the property, known as *additivity*, that $N(M_1 \cup M_2) = N(M_1) + N(M_2) - N(M_1 \cap M_2)$ for any compact subanalytic sets M_1 and M_2 [3, Theorem 4.2].

3.1 The normal cycle of a smooth surface

In their work on the curvature measures of smooth surfaces M without boundary, Cohen Steiner and Morvan used the fact that $M = \partial V$ is the boundary of a three dimensional region $V \subset \mathbb{R}^3$. In this case, the normal cycle of M is given by the $N(M) = i_{+\#}[M]$. Here $i_+ : M \to T\mathbb{R}^3$ is defined by $i_+(p) = (p, \vec{U}(p))$, where $\vec{U}$ is the positive unit normal vector field to M. To distinguish this definition of the normal cycle from the one we will use in this thesis, we use notation $N(V)$ for $i_{+\#}[M]$ [10].

We want all normal cycles to be cycles, i.e. to have zero boundary, so when we consider the normal cycle of a smooth surface with boundary we must consider both the positive and negative unit normal vectors to the surface.

Definition. Suppose M is a smooth surface with boundary. The *normal cycle of M above the interior of M* is the integral current $N_{\mathrm{int}\,M}(M)$ defined by $N_{\mathrm{int}\,M}(M) = i_{+\#}[M] - i_{-\#}[M]$, where $i_+, i_- : M \to T\mathbb{R}^3$ are the maps $i_+(p) = (p, +\vec{U}(p))$ and $i_-(p) = (p, -\vec{U}(p))$. Let $(\vec{T}, \vec{V}, \vec{U})$ be the Darboux frame field along the smooth boundary of M. The *normal cycle of M above the boundary of M* is the integral current $N_{\partial M}(M)$ defined by $N_{\partial M}(M) = i_{\partial\#}([\partial M] \times [0, \pi])$ where $i_\partial : \partial M \times [0, \pi] \to T\mathbb{R}^3$ is the map $i_\partial(p, \theta) = (p, -\vec{U}(p)\cos\theta - \vec{V}(p)\sin\theta)$. Then the *normal cycle of M* is the integral current $N(M)$ defined by $N(M) = N_{\mathrm{int}\,M}(M) + N_{\partial M}(M)$.

Geometrically, $N_{\mathrm{int}\,M}(M)$ is an integral current associated with both the positive and negative unit normal bundles on M, and $N_{\partial M}(M)$ is an integral current associated with a bundle of outward half-circular arcs of unit vectors along ∂M. The orientation of these sets associated with $N(M)$ are chosen to be consistent with the Joe Fu's definition of a normal cycle and so that $N(M)$ is a cycle:

$$\begin{aligned}
\partial N(M) &= \partial i_{+\#}[M] - \partial i_{-\#}[M] + \partial i_{\partial\#}([\partial M] \times [0, \pi]) \\
&= i_{+\#}[(\partial M)] - i_{-\#}[(\partial M)] + ([\{(p, -\vec{U}(p)) : p \in \partial M\}] - [\{(p, \vec{U}(p)) : p \in \partial M\}]) \\
&= [\{(p, \vec{U}(p)) : p \in \partial M\}] - [\{(p, -\vec{U}(p)) : p \in \partial M\}] + [\{(p, -\vec{U}(p)) : p \in \partial M\}] \\
&\quad -[\{(p, \vec{U}(p)) : p \in \partial M\}] \\
&= 0.
\end{aligned}$$

We can visualize a normal cycle using the map $(p, n) \mapsto p + n$ for $(p, n) \in T\mathbb{R}^n$. In Figure 1, we see that the normal cycle of a disk looks like a shell surrounding the disk.

Note that for a surface M without boundary that $N(M) \neq N(V)$ where $M = \partial V$. For example, if M is a sphere and V is the inside of the sphere, then $N(V)$ is the integral current associated with the outward unit normal bundle of the sphere, whereas $N(M)$ is an integral current associated with the union of the outward and inward unit normal bundles of the sphere.

3.2 The normal cycle of a triangulated surface

We shall consider two approaches to describing the normal cycle of a triangulated surface T. The first approach is to determine the normal cycles of the vertices, edges, and faces of T, which are themselves subanalytic sets, and then use additivity to add the normal cycles of the vertices, edges, and faces of T together to obtain $N(T)$. The normal cycle $N(v)$ of a vertex v is the integral current associated with the sphere of unit vectors in $T_v\mathbb{R}^3$. The normal cycle $N(e)$ of an edge e is the cylindrical integral current associated with unit normal vectors along

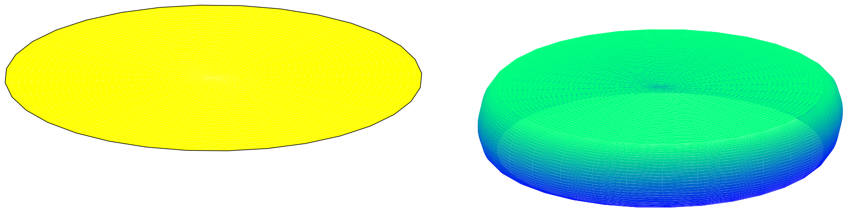
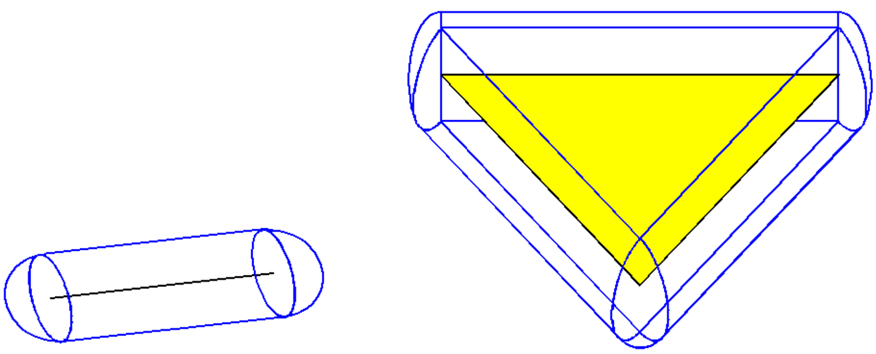


Figure 1: The normal cycle of a circular disk can be represented in $\mathbb{R}^3$ as the set of points at distance ϵ from the disk.

e that are perpendicular to e plus integral currents associated with hemispherical caps of unit vectors at the vertices of e, shown in Figure 2(a). The normal cycle $N(f)$ of a face f is the sum of the following integral currents: the currents associated with the planes of positive and negative unit normal bundles to f, the currents associated with the half-cylinder of unit vectors along each edge of f that are perpendicular to that edge, and the currents supported on the sphere of unit vectors at each vertex v of f associated with a spherical polygon, shown in Figure 2(b). These spherical polygons are 2-gons which have precisely two vertices at the two unit normal vectors to f and boundary curves along the great circles perpendicular to the edges incident at v. The orientation of the normal cycles of the vertices, edges, and faces of T is given by the outward unit normal vector to the spherical parts being the positive unit normal vector and the orientation on the cylindrical and planar parts being chosen so that the normal cycles have zero boundary. Note that in Figure 2 we are visualizing the normal cycle using the map $(p, n) \mapsto p + n$ for $(p, n) \in T\mathbb{R}^3$.

Figure 2: The normal cycle of a face and an edge.



(a) Normal cycle of an edge.

(b) Normal cycle of a face.

We then define the normal cycle of T by applying the additivity property of normal cycles using the generalized Inclusion-Exclusion Principle.

Definition. For a triangulated surface T, the *normal cycle of T* is the integral current defined by

$$N(T) = N\left(\bigcup_{i=1}^{n} f_i\right) = \sum_{k=1}^{n}(-1)^{k-1} \sum_{1\leq\sigma(1)<\sigma(2)<...<\sigma(k)\leq n} N\left(\bigcap_{i=1}^{k} f_{\sigma(i)}\right) \tag{2}$$

where T has n faces f_i and each $\sigma : \{1, 2, ..., k\} \to \{1, 2, ..., n\}$ gives the indices of k faces of T [10].

Theorem 1. *For a triangulated surface* $T \subseteq \mathbb{R}^3$*,*

$$N(T) = \sum_{f\subset T \text{ a face}} N(f) - \sum_{e\subset \operatorname{int} T \text{ an edge}} N(e) + \sum_{v\in \operatorname{int} T \text{ a vertex}} N(v),$$

Proof. Since any intersection of the faces of T must be a face, edge, or vertex of T, (2) is equivalent to

$$N(M) = \sum_{f\subset T \text{ a face}} a(f)N(f) + \sum_{e\subset \operatorname{int} T \text{ an edge}} b(e)N(e) + \sum_{v\in \operatorname{int} T \text{ a vertex}} c(v)N(v). \tag{3}$$

for some integers $a(f)$, $b(e)$, and $c(v)$ depended on f, e, and v, respectively. Thus to proof the claim, we must determine the $a(f)$, $b(e)$, and $c(v)$ using combinatorics.

For every face f, $N(f)$ occurs exactly once in (2) with multiplicity $+1$, thus $a(f) = 1$. For every edge $e \subset \operatorname{int} T$, $N(e)$ occurs exactly once in (2) with multiplicity -1 as the intersection of two faces. So if $e \subset \operatorname{int} T$, $b(e) = -1$. No edge $e \subset \partial T$ can be formed as the intersection of faces and thus if $e \subset \partial M$, $b(e) = 0$.

For a vertex $v \in \operatorname{int} T$, suppose there m faces of T incident at v. Let $\operatorname{Star}(v, T)$ denote the union of v and the interiors of the faces and edges incident to v. The only terms in (2) that contribute to $N(v)$ belong to the $\operatorname{Star}(v, T)$, so we will restrict our attention to $\operatorname{Star}(v, T)$. There are $\binom{m}{k}$ ways to intersect k faces in $\operatorname{Star}(v, T)$. Thus by counting the multiplicities of faces, edges, and vertices in $\operatorname{Star}(v, T)$ and equating (2) and (3),

$$\begin{aligned} 1 &= \sum_{k=1}^{m}(-1)^{k-1}\binom{m}{k} = \sum_{f\subset Star(vT) \text{ a face}} a(f) + \sum_{e\subset Star(v,T) \text{ an edge}} b(e) + c(v) \\ &= \#\{\text{faces } f \text{ in } Star(v,T)\} - \#\{\text{edges } e \text{ in } Star(v,T)\} + c(v) \\ &= m - m + c(v) = c(v). \end{aligned} \tag{4}$$

Thus if $v \in \partial T$, $c(v) = 1$. If $v \notin \partial T$ and there m faces of T incident at v, the number of edges in $\operatorname{int} T$ incident to v is $m-1$. The counts in (4) are otherwise unchanged and it follows that if $v \notin \partial T$, $c(v) = 0$.

Therefore $a(f) = 1$ if f is a face of T, $b(e) = -1$ if $e \subset \operatorname{int} T$ is an edge of T, $c(v) = 1$ if $v \in \operatorname{int} T$ is a face of T, and $a(f)$, $b(e)$, and $c(v)$ are all zero otherwise. Thus (3) is equivalent to the claim. □

The following corollary can be applied to most of the measures that we will derive and is stated in its most general form. The corollary is a direct consequence of the properties of integral currents and Theorem 1.

Corollary 1. *Let T be a triangulated surface and ω be a 2-form. If $S = T$ or S is a face, edge, or vertex of T, define the measure ϕ_S by $\phi_S(B) = \frac{1}{2}\langle N(S)\llcorner\pi^{-1}(B), \omega\rangle$ for any Borel set B. Then*

$$\phi_T = \sum_{f\subset T \text{ a face}} \phi_f - \sum_{e\subset \operatorname{int} T \text{ an edge}} \phi_e + \sum_{v\in \operatorname{int} T \text{ a vertex}} \phi_v. \tag{5}$$

The other approach to describing $N(T)$ is to write $N(T)$ as the sum of planar, cylindrical, and spherical integral currents supported above the faces, edges, and vertices of T, respectively. We used a similar approach for describing the normal cycle of a smooth surface, writing the normal cycle of a smooth surface M as the sum of an integral current supported above the interior of M and an integral current supported above the boundary of M.

Theorem 2. *We can write the normal cycle of T as*

$$N(T) = \sum_{f\subset T \text{ a face}} N_f(T) + \sum_{e\subset T \text{ an edge}} N_e(T) + \sum_{v\in T \text{ a vertex}} N_v(T),$$

where $N_f(T) = N(T)\llcorner\pi^{-1}(\operatorname{int} f)$, $N_e(T) = N(T)\llcorner\pi^{-1}(\operatorname{int} e)$, and $N_v(T) = N(T)\llcorner\pi^{-1}(\{v\})$.

Proof. Since T is a disjoint union of the interiors of its faces, interior of its edges, and its vertices,

$$\begin{aligned}
N(T) &= N(T)\llcorner\pi^{-1}(T) \\
&= N(T)\llcorner\pi^{-1}\left(\bigcup_{f\subset T \text{ a face}} \operatorname{int} f \;\cup \bigcup_{e\subset T \text{ an edge}} \operatorname{int} e \;\cup \bigcup_{v\in T \text{ a vertex}} \{v\}\right) \\
&= N(T)\llcorner\left(\bigcup_{f\subset T \text{ a face}} \pi^{-1}(\operatorname{int} f) \;\cup \bigcup_{e\subset T \text{ an edge}} \pi^{-1}(\operatorname{int} e) \;\cup \bigcup_{v\in T \text{ a vertex}} \pi^{-1}(\{v\})\right) \\
&= \sum_{f\subset T \text{ a face}} N(T)\llcorner\pi^{-1}(\operatorname{int} f) + \sum_{e\subset T \text{ an edge}} N(T)\llcorner\pi^{-1}(\operatorname{int} e) + \sum_{v\in T \text{ a vertex}} N(T)\llcorner\pi^{-1}(\{v\}) \\
&= \sum_{f\subset T \text{ a face}} N_f(T) + \sum_{e\subset T \text{ an edge}} N_e(T) + \sum_{v\in T \text{ a vertex}} N_v(T).
\end{aligned}$$

□

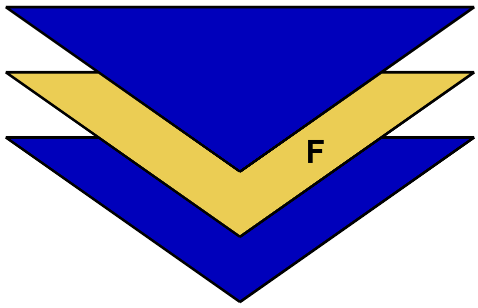
Note that $N_f(T)$ and $N(f)$ denote two different integral currents. In particular $N(f)$ is cycle where as $N_f(T)$ is not a cycle. Similarly, $N_e(T) \neq N(e)$ and $N_v(T) \neq N(v)$.

Given a face f of T, $N_f(T)$ is called the *normal cycle of T above f* and is the integral current associated with the positively oriented unit normal vectors to f with the orientation induced by f minus the integral current associated with the negatively oriented unit normal vectors to f with the orientation induced by f. Given an edge e of T, $N_e(T)$ is called the *normal cycle of T above e* and is given by $N_e(T) = [C_e^+]-[C_e^-]$ where C_e^+ and C_e^- are compact smooth manifolds defined as follows (see Figure 3). The manifold C_e^+ is the cylindrical subset consisting of $(p, n) \in T\mathbb{R}^3$ with $p \in e$ and n lying on the shortest circular arc between the positively oriented unit normal vectors to the faces incident to e. Similarly, the manifold C_e^- is the cylindrical subset consisting of $(p, n) \in T\mathbb{R}^3$ with $p \in e$ and n lying on the shortest circular arc between the negatively oriented unit normal vectors to the faces incident to e. Given a vertex v of T, $N_v(T)$ is called the *normal cycle of T above v*. The normal cycle of T above v is supported on the sphere of unit vectors in $T_v\mathbb{R}^3$. Since $N_v(T)$ is a complicated integral current, we shall not provide an explicit description of this integral current. Details are given in [10].

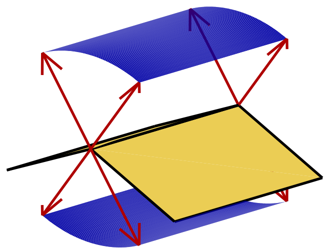
4 Gauss curvature measure

We will generalize the Gauss curvature measure studied by Cohen-Steiner to smooth and triangulated surfaces with boundary [10]. Recall that the Gauss curvature $K(p)$ at a point

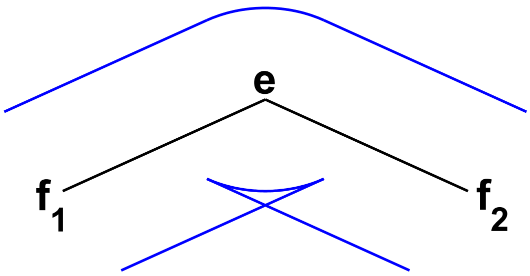
Figure 3: The normal cycle of above a face, edge, and vertex.



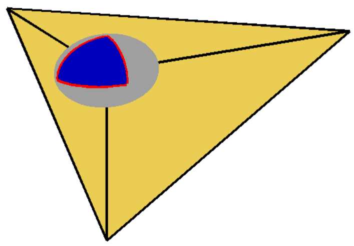
(a) Normal cycle above a face.



(b) Normal cycle above an edge (3D view).



(c) Normal cycle above an edge (front view).



(d) Normal cycle above a vertex.

p in a smooth surface M equals the product of the principal curvatures of M at p. Our goal is to define the Gauss curvature measure ϕ^G_M of a surface M using the normal cycle of M in such a way that for the special case where M is a smooth surface M without boundary, for any Borel set $B \subseteq \mathbb{R}^3$,

$$\phi^G_M(B) = \int_{M\cap B} K dA.$$

Definition. Let $p \in \mathbb{R}^3$ and $n \in T_p\mathbb{R}^3$ be a unit vector and let $(\vec{u}_1, \vec{u}_2, n)$ be a positively oriented orthonormal frame. The Gauss curvature 2-form ω^G on $T\mathbb{R}^3$ is defined by

$$\omega^G_{(p,n)} = (0, \vec{u}_1) \wedge (0, \vec{u}_2).$$

Note that $\omega^G_{(p,n)}$ is independent of the choice of frame $(\vec{u}_1, \vec{u}_2, n)$ [10].

Let $M \subseteq \mathbb{R}^3$ be a smooth or triangulated surface. Then the Gauss curvature measure ϕ^G_M is defined by

$$\phi^G_M(B) = \tfrac{1}{2}\langle N(M) \llcorner \pi^{-1}(B), \omega^G\rangle.$$

for all Borel sets $B \subseteq \mathbb{R}^3$.

4.1 Smooth surfaces with boundary

Recall that the arclength form on a smooth curve $\gamma \subseteq \mathbb{R}^3$ is the 1-form ds such that $ds(\vec{T}) = 1$ for the positive unit tangent vector $\vec{T}$ to γ. The area form on a smooth surface M is the 2-form dA such that $dA(\vec{v}_1, \vec{v}_2) = 1$ for any positively oriented orthonormal frame of vectors $(\vec{v}_1, \vec{v}_2)$.

Theorem 3. *For a smooth surface $M \subseteq \mathbb{R}^3$ with a smooth boundary,*

$$\phi^G_M = K dA \wedge [M] + \kappa_g ds \wedge [\partial M].$$

where κ_g is the geodesic curvature of ∂M.

Remark. In the above equation, we regard zero-dimensional currents as measures in the following sense: if S is a zero dimensional current, then $S(B) = S(\chi_B)$ for all Borel sets B, where χ_B is the characteristic function of B.

Proof. Let $B \subseteq \mathbb{R}^3$ be a Borel set. Let $\vec{U}$ be the positively oriented unit normal vector field on M and $(\vec{T}, \vec{V}, \vec{U})$ be the Darboux frame field along ∂M. Recall that

$$N(M) = i_{+\#}[M] - i_{-\#}[M] + i_{\partial\#}([\partial M] \times [0, \pi]) \tag{6}$$

where

$$\begin{aligned} i_+(p) &= (p, \vec{U}(p)), \\ i_-(p) &= (p, -\vec{U}(p)), \\ i_\partial(p, \theta) &= (p, -\vec{U}(p)\cos\theta - \vec{V}(p)\sin\theta). \end{aligned}$$

Recall from multivariable calculus that

$$\langle i_{+\#}[M] \llcorner \pi^{-1}(B), \omega^G\rangle = \int_{i_+(M\cap B)} \omega^G = \int_{M\cap B} \omega^G\left(\nabla_{\vec{u}_1} i_+, \nabla_{\vec{u}_2} i_+\right) dA$$

where $\nabla_{\vec{Y}} X = (\vec{Y}(X_1), \vec{Y}(X_2), \vec{Y}(X_3))$ is the covariant derivative of $\vec{X} = (X_1, X_2, X_3)$ in the direction $\vec{Y}$. For $p \in \operatorname{int} M$, let $(\vec{u}_1, \vec{u}_2, \vec{U}(p))$ be a positively oriented orthonormal frame. Recall that the second fundamental form h is defined by $h(\vec{X}, \vec{Y}) = -\nabla_{\vec{X}} \vec{U} \bullet \vec{Y}$ for vector fields $\vec{X}, \vec{Y}$. Then

$$\omega^G_{(p,\vec{U}(p))}(\nabla_{\vec{u}_1} i_+, \nabla_{\vec{u}_2} i_+) = \begin{vmatrix} \vec{u}_1 \bullet \nabla_{\vec{u}_1} \vec{U} & \vec{u}_2 \bullet \nabla_{\vec{u}_1} \vec{U} \\ \vec{u}_1 \bullet \nabla_{\vec{u}_2} \vec{U} & \vec{u}_2 \bullet \nabla_{\vec{u}_2} \vec{U} \end{vmatrix} = \begin{vmatrix} -h_p(\vec{u}_1, \vec{u}_1) & -h_p(\vec{u}_2, \vec{u}_1) \\ -h_p(\vec{u}_1, \vec{u}_2) & -h_p(\vec{u}_2, \vec{u}_2) \end{vmatrix} = K$$

since Gauss curvature is the determinant of the Second Fundamental Form. Similarly $\omega^G_{(p,\vec{U}(p))}(\nabla_{\vec{u}_1} i_-, \nabla_{\vec{u}_2} i_-) = -K$. So

$$\langle (i_{+\#}[M] - i_{-\#}[M]) \llcorner \pi^{-1}(B), \omega^G \rangle = \int_M 2K dA = 2K dA \wedge [M](B). \tag{7}$$

Next recall from multivariable calculus that

$$\langle i_{\partial\#}([\partial M] \times [0,\pi]) \llcorner \pi^{-1}(B), \omega^G \rangle = \int_{i_\partial(\partial M \cap B \times [0,\pi])} \omega^G = \int_{\partial M \cap B} \int_0^\pi \omega^G \left(\tfrac{di_\partial}{ds}, \tfrac{di_\partial}{d\theta} \right) d\theta ds$$

where we regard $p(s)$ as a unit speed parameterization of the boundary of M. For $p \in \partial M$ and $\theta \in [0, \pi]$, $(\vec{u}_1, \vec{u}_2, n) = (T, U \sin\theta - V \cos\theta, -U \cos\theta - V \sin\theta)$ is an orthonormal frame on ∂M. We compute

$$\begin{aligned} \tfrac{dn}{ds} &= [h(\vec{T}, \vec{T})\vec{T} + h(\vec{T}, \vec{V})\vec{V}] \cos\theta + [\kappa_g \vec{T} - h(\vec{T}, \vec{V})\vec{U}] \sin\theta \\ &= [h(\vec{T}, \vec{T}) \cos\theta + \kappa_g \sin\theta] \vec{u}_1 + h(\vec{T}, \vec{V}) \vec{u}_2, \\ \tfrac{dn}{d\theta} &= \vec{V} \sin\theta - \vec{U} \cos\theta = \vec{u}_2, \end{aligned}$$

so

$$\omega^G_{(p,n)} \left(\tfrac{di_\partial}{ds}, \tfrac{di_\partial}{d\theta} \right) = \begin{vmatrix} \vec{u}_1 \bullet \nabla_{\vec{u}_1} n & \vec{u}_2 \bullet \nabla_{\vec{u}_1} n \\ \vec{u}_1 \bullet \nabla_{\vec{u}_2} n & \vec{u}_2 \bullet \nabla_{\vec{u}_2} n \end{vmatrix} = \kappa_g \cos\theta + h(\vec{T}, \vec{T}) \sin\theta.$$

Hence

$$\begin{aligned} \langle i_{\partial\#}([\partial M] \times [0,\pi]) \llcorner \pi^{-1}(B), \omega^G \rangle &= \int_{\partial M \cap B} \int_0^\pi (h(\vec{T}, \vec{T}) \cos\theta + \kappa_g \sin\theta) d\theta ds \\ &= \int_{\partial M \cap B} 2\kappa_g ds = 2\kappa_g ds \wedge [\partial M](B). \end{aligned} \tag{8}$$

The result follows from (6), (7), and (8). □

4.2 Triangulated surfaces with boundary

Theorem 4. *For a triangulated surface $T \subseteq \mathbb{R}^3$,*

$$\phi^G_T = \sum_{v \in \operatorname{int} T \text{ a vertex}} \alpha(v)[v] + \sum_{v \in \partial T \text{ a vertex}} \beta(v)[v].$$

where $\alpha(v)$ equals 2π minus the sum of the angles incident at v and $\beta(v)$ equals π minus the sum of the angles incident at v.

Proof. Consider the normal cycle above the faces, edges, and vertices of T. Let f be a face of T and $(p, n) \in \operatorname{spt} N_f(T)$. If $(\vec{u}_1, \vec{u}_2, n)$ is an orthonormal frame, then the vectors $(\vec{u}_1, 0)$ and $(\vec{u}_2, 0)$ span the tangent plane to $\operatorname{spt} N_f(T)$. Since $\omega^G_{(p,n)}((\vec{u}_1, 0), (\vec{u}_2, 0)) = 0$, $\omega^G|_{\operatorname{spt} N_f(T)} = 0$. Similarly, for an edge e of T, $\omega^G|_{\operatorname{spt} N_e(T)} = 0$, and for a vertex v of T, $\omega^G|_{\operatorname{spt} N_v(T)}$ is the area form for the sphere of unit vectors in $T_v\mathbb{R}^3$. Thus $\omega^G|_{\operatorname{spt} N(T)}$ is supported on above the vertices of T.

By Corollary 1, we can write the normal cycle as

$$\phi^G_T = \sum_{f \subset T \text{ a face}} \phi^G_f - \sum_{e \subset \operatorname{int} T \text{ an edge}} \phi^G_e + \sum_{v \in \operatorname{int} T \text{ a vertex}} \phi^G_v. \tag{9}$$

For a vertex v of T, $N(v)$ is the current associated with the sphere of unit vectors in $T_v\mathbb{R}^3$, so $\phi^G_v = 2\pi[v]$. For an edge e in the interior of T joining two vertices v_1 and v_2, the normal cycle of e above either vertex is a hemisphere of unit vectors and thus $\phi^G_e = \pi[v_1] + \pi[v_2]$. For a triangular face f with edges v_1, v_2, and v_3, the normal cycle of f above v_i is an integral current associated with a spherical 2-gon joining the unit normal vectors to f. If ϑ_i is the angle between the edges of f incident at v_i, the angular width of this 2-gon is $\pi - \vartheta_i$ along the great circle perpendicular to the unit normal vectors of f. So $\phi^G_f = \sum_{i=1}^{3}(\pi - \vartheta_i)[v_i]$.

Now fix an vertex v of M. Suppose there are m faces f_i incident to v at T and the angles between the two edges of f_i incident at v is θ_i. If $v \in \operatorname{int} M$, by (9),

$$\phi^G_T \llcorner \{v\} = \sum_{i=1}^{m} (\pi - \theta_i)[v] - m\pi[v] + 2\pi[v] = \left(2\pi - \sum_{i=1}^{m} \theta_i\right)[v] = \alpha(v)[v],$$

while if $v \in \partial M$,

$$\phi^G_T \llcorner \{v\} = \sum_{i=1}^{m} (\pi - \theta_i)[v] - (m-1)\pi[v] + 0 = \left(\pi - \sum_{i=1}^{m} \theta_i\right)[v] = \beta(v)[v].$$

□

4.3 Gauss-Bonnet Formula

An important result from classical differential geometry is the Gauss-Bonnet formula.

Theorem 5. *(Gauss-Bonnet Formula) For a smooth surface $M \subseteq \mathbb{R}^3$ with a smooth boundary,*

$$2\pi\chi(M) = \int\int_M K dA + \int_{\partial M} \kappa_g ds,$$

where $\chi(M)$ is the Euler characteristic of M.

In terms of curvature measures, we can express the Gauss-Bonnet formula as $\phi^G_M(M) = 2\pi\chi(M)$. We will prove this fact for triangulated surfaces.

Theorem 6. *For a triangulated surface $T \subseteq \mathbb{R}^3$ with a smooth boundary,*

$$\phi^G_T(T) = 2\pi\chi(T).$$

Proof. In the special case that T is a triangle, $\phi^G_T(T)$ equals 3π minus the sum of the interior angles of T. Since the sum of the interior angles of a triangle is π, $\phi^G_T(T) = 2\pi$. Similarly, for any edge e of T, $\phi^G_e(T) = 2\pi$, and for any vertex v of T, $\phi^G_v(T) = 2\pi$. So by (9),

$$\begin{aligned} \phi^G_T(T) &= \sum_{f \subset T \text{ a face}} \phi^G_f(T) - \sum_{e \subset \operatorname{int} T \text{ an edge}} \phi^G_e(T) + \sum_{v \in \operatorname{int} T \text{ a vertex}} \phi^G_v(T) \\ &= 2\pi \cdot \#\{\text{faces of } T\} - 2\pi \cdot \#\{\text{edges in } \operatorname{int} T\} + 2\pi \cdot \#\{\text{vertices in } \operatorname{int} T\}. \end{aligned}$$

Since $\#\{\text{edges in } \partial T\} = \#\{\text{vertices in } \partial T\}$,

$$\phi^G_M(M) = 2\pi \cdot \#\{\text{faces of } T\} - 2\pi \cdot \#\{\text{edges of } T\} + 2\pi \cdot \#\{\text{vertices of } T\} = 2\pi\chi(M),$$

proving the claim. □

5 Mean curvature vector

Recall that the mean curvature $H(p)$ at a point p on a smooth surface M is the average of the principal curvatures of M at p. We would like to derive a curvature measure ϕ^H_M for smooth and triangulated surfaces with boundary such that in the special case that M is a smooth surface without boundary,

$$\phi^H_M(B) = \int_{M \cap B} 2H dA.$$

In [10], such a measure is defined when M is a smooth surface without boundary by

$$\phi^H_M(B) = -\langle i_{+\#}[M] \llcorner \pi^{-1}(B), \omega^H \rangle,$$

where the 2-form ω^H is defined so that if $p \in \mathbb{R}^3$, $n \in T_p\mathbb{R}^3$ is a unit vector, and $(\vec{u}_1, \vec{u}_2, n)$ is a positively orthonormal frame, then

$$\omega^H_{(p,n)} = (\vec{u}_1, 0) \wedge (0, \vec{u}_2) + (0, \vec{u}_1) \wedge (\vec{u}_2, 0).$$

Note that $\omega^H_{(p,n)}$ is independent of the choice of frame $(\vec{u}_1, \vec{u}_2, n)$. (Also note that [10] defines mean curvature as -1 times the sum of the principal curvatures.)

An obvious way to extend ϕ^H_M to surfaces with boundary is by defining

$$\phi^H_M(B) = \tfrac{1}{2}\langle N(M) \llcorner \pi^{-1}(B), -\omega^H \rangle.$$

However, for a smooth surface M without boundary,

$$\begin{aligned} \phi^H_M(B) &= \tfrac{1}{2}\langle N(M) \llcorner \pi^{-1}(B), -\omega^H \rangle = \tfrac{1}{2}\langle i_{+\#}[M] \llcorner \pi^{-1}(B), -\omega^H \rangle - \tfrac{1}{2}\langle i_{-\#}[M] \llcorner \pi^{-1}(B), -\omega^H \rangle \\ &= \tfrac{1}{2}\int_{M \cap B} 2H dA - \tfrac{1}{2}\int_{M \cap B} 2H dA = 0. \end{aligned} \tag{10}$$

One way to try to get around this cancellation would be to define a normal cycle $N^+(M)$ of M such that $N^+(M) = i_{+\#}[M]$ for a smooth surface M without boundary. However, if M is a surface with boundary, we can not define such a normal cycle $N^+(M)$ in such a way that $\partial N^+(M) = 0$. Another approach is to instead define a mean curvature *vector* measure; that is define a measure $\phi^{H\vec{U}}_M$ for a smooth or triangulated surface M such that for the special case where M is a smooth surface without boundary, for any Borel set $B \subseteq \mathbb{R}^3$,

$$\phi^{H\vec{U}}_M(B) = \int_{M \cap B} 2H\vec{U} dA.$$

This approach is motivated by the fact that the cancellation in (10) occurs due to orientation issues: by reversing the direction of $\vec{U}$ we reverse the sign of H and the orientation of M and thus $\langle i_{+\#}[M] \llcorner \pi^{-1}(B), \omega^H\rangle = \langle i_{-\#}[M] \llcorner \pi^{-1}(B), \omega^H\rangle$. We did not have this problem with Gauss curvature since the Gauss curvature is independent of the orientation of M. The mean curvature vector $H\vec{U}$ is also independent of the orientation of M.

Definition. Let $M \subseteq \mathbb{R}^3$ be a smooth or triangulated surface. The mean curvature vector 2-form $\omega^{H\vec{U}}$ is a vector-valued 2-from on $T\mathbb{R}^3$ defined by $\omega^{H\vec{U}}_{(p,n)} = -n\omega^H_{(p,n)}$ for $p \in \mathbb{R}^3$ and $n \in T_p\mathbb{R}^3$. The mean curvature vector measure $\phi^{H\vec{U}}_M$ is defined by

$$\phi^{H\vec{U}}_M(B) = \tfrac{1}{2}\langle N(M) \llcorner \pi^{-1}(B), \omega^{H\vec{U}}\rangle,$$

for all Borel sets $B \subseteq \mathbb{R}^3$.

Note that $\omega^{H\vec{U}}$ is a vector valued measure, i.e. for every Borel set $B \subseteq \mathbb{R}^3$, $\omega^{H\vec{U}}(B)$ is a vector.

5.1 Smooth surfaces with boundary

Theorem 7. *Let $M \subseteq \mathbb{R}^3$ be an oriented smooth surface with a smooth boundary and let $(\vec{T}, \vec{V}, \vec{U})$ be a Darboux frame on the boundary of M. Then*

$$\phi^{H\vec{U}}_M = 2H\vec{U} dA \wedge [M] + \vec{V} ds \wedge [\partial M].$$

Proof. We will modify the prove of the Gauss curvature measure theorem for smooth surfaces. Let $B \subseteq \mathbb{R}^3$ be a Borel set. Recall that

$$N(M) = i_{+\#}[M] - i_{-\#}[M] + i_{\partial\#}([\partial M] \times [0, \pi]). \tag{11}$$

For $p \in \operatorname{int} M$, let $\vec{e}_1, \vec{e}_2$ be the principal directions[1] with corresponding principal curvatures κ_1, κ_2 and observe that $(\vec{e}_1, \vec{e}_2, \vec{U}(p))$ is an orthonormal frame. Without loss of generality, suppose $(\vec{e}_1, \vec{e}_2, \vec{U}(p))$ is a positively oriented frame. Recall from multivariable calculus that

$$\langle i_{+\#}[M] \llcorner \pi^{-1}(B), \omega^{H\vec{U}}\rangle = \int_{M\cap B} \omega^{H\vec{U}}\left(\nabla_{\vec{e}_1} i_+, \nabla_{\vec{e}_2} i_+\right) dA.$$

We compute

$$\begin{aligned} \omega^{H\vec{U}}_{(p,\vec{U}(p))}(\nabla_{\vec{e}_1} i_+, \nabla_{\vec{e}_2} i_+) &= -\vec{U}(p) \begin{vmatrix} \vec{e}_1 \bullet \vec{e}_1 & \vec{e}_2 \bullet \nabla_{\vec{e}_1}\vec{U} \\ \vec{e}_1 \bullet \vec{e}_2 & \vec{e}_2 \bullet \nabla_{\vec{e}_2}\vec{U} \end{vmatrix} - \vec{U}(p) \begin{vmatrix} \vec{e}_1 \bullet \nabla_{\vec{e}_1}\vec{U} & \vec{e}_2 \bullet \vec{e}_1 \\ \vec{e}_1 \bullet \nabla_{\vec{e}_2}\vec{U} & \vec{e}_2 \bullet \vec{e}_2 \end{vmatrix} \\ &= -\vec{U}(p) \begin{vmatrix} 1 & 0 \\ 0 & -\kappa_2 \end{vmatrix} - \vec{U}(p) \begin{vmatrix} -\kappa_1 & 0 \\ 0 & 1 \end{vmatrix} = \kappa_1 \vec{U}(p) + \kappa_2 \vec{U}(p) \\ &= 2H\vec{U}(p). \end{aligned}$$

Similarly $\omega^{H\vec{U}}_{(p,-\vec{U}(p))}(\nabla_{\vec{e}_1} i_-, \nabla_{\vec{e}_2} i_-) = -2H\vec{U}(p)$. So

$$\langle (i_{+\#}[M] - i_{-\#}[M]) \llcorner \pi^{-1}(B), \omega^{H\vec{U}}\rangle = \int_M 4H\vec{U}(p) dA = 4H\vec{U}(p) dA \wedge [M](B). \tag{12}$$

[1]If p is an umbilic point, i.e. $\kappa_1 = \kappa_2$, we choose $\vec{e}_1, \vec{e}_2$ to be any pair of orthogonal unit vectors in T_pM.

Next recall that

$$\langle i_{\partial\#}([\partial M]\times[0,\pi])\llcorner\pi^{-1}(B),\omega^{H\vec{U}}\rangle = \int_{\partial M\cap B}\int_0^\pi \omega^{H\vec{U}}\left(\tfrac{di_\partial}{ds},\tfrac{di_\partial}{d\theta}\right)d\theta ds$$

where we regard $p(s)$ as a unit speed parameterization of the boundary of M. For $p\in\partial M$, $(\vec{u}_1,\vec{u}_2,n)=(\vec{T},\vec{U}\sin\theta-\vec{V}\cos\theta,-\vec{U}\cos\theta-\vec{V}\sin\theta)$ is an orthonormal frame on ∂M. We compute

$$\begin{aligned}\tfrac{dn}{ds} &= [h(\vec{T},\vec{T})\cos\theta+\kappa_g\sin\theta]\vec{u}_1-h(\vec{T},\vec{V})\vec{u}_2,\\ \tfrac{dn}{d\theta} &= \vec{U}\sin\theta-\vec{V}\cos\theta=\vec{u}_2,\end{aligned}$$

so

$$\begin{aligned}\omega^{H\vec{U}}\left(\tfrac{di_\partial}{ds},\tfrac{di_\partial}{d\theta}\right) &= -n\begin{vmatrix}\vec{u}_1\bullet\vec{T} & \vec{u}_2\bullet\frac{dn}{ds}\\ \vec{u}_1\bullet 0 & \vec{u}_2\bullet\frac{dn}{d\theta}\end{vmatrix}-n\begin{vmatrix}\vec{u}_1\bullet\frac{dn}{ds} & \vec{u}_2\bullet\vec{T}\\ \vec{u}_1\bullet\frac{dn}{d\theta} & \vec{u}_2\bullet 0\end{vmatrix}\\ &= -n\begin{vmatrix}1 & \vec{u}_2\bullet\frac{dn}{ds}\\ 0 & \vec{u}_2\bullet\vec{u}_2\end{vmatrix}-n\begin{vmatrix}\vec{u}_1\bullet\frac{dn}{ds} & 0\\ \vec{u}_1\bullet\vec{u}_2 & 0\end{vmatrix}=-n.\end{aligned}$$

Hence

$$\begin{aligned}\langle i_{\partial\#}([\partial M]\times[0,\pi])\llcorner\pi^{-1}(B),\omega^{H\vec{U}}\rangle &= \int_{\partial M\cap B}\int_0^\pi -n d\theta ds\\ &= \int_{\partial M\cap B}\int_0^\pi(\vec{U}\cos\theta+\vec{V}\sin\theta)d\theta ds\\ &= \int_{\partial M\cap B}2\vec{V}ds=2\vec{V}ds\wedge[\partial M](B).\end{aligned}\tag{13}$$

The result follows from (11), (12), and (13). □

One advantage of the mean curvature vector measure is that we can use it to approximate both mean curvature and the unit normal vector field to a surface. We can approximate the mean curvature at a point p by $\phi_M^H(B_p)/\operatorname{Area}(B_p)$ and we can approximate the unit normal vector field to a surface M at p by the vector $\phi^{H\vec{U}}(B_p)/\|\phi^{H\vec{U}}(B_p)\|$, where B_p is a small open neighborhood of p. This method for approximating unit normal vectors assumes that $\phi^{H\vec{U}}(B_p)\neq 0$. In the case that $\phi^{H\vec{U}}(B_p)=0$ for a smooth surface M, we have

$$2H(p)\vec{U}(p)\approx\frac{\phi_M^H(B_p)}{\operatorname{Area}(B_p)}=0\Rightarrow H(p)\approx 0.$$

A good example of this behavior is the saddle surface $z=x^2-y^2$ where p is the origin. We conclude that $\phi_M^H(B)=0$ is often an indication that the surface (or surface being approximated, in the case of triangulated surfaces), has zero mean curvature. Thus one should not use this method for computing unit normal vectors in this case.

5.2 Triangulated surfaces with boundary

Definition. Let $T\subseteq\mathbb{R}^3$ be a triangulated surface and e be an edge of T. If $e\subset\operatorname{int}T$, let $\vec{U}_1$ and $\vec{U}_2$ be the positively oriented unit normal vectors to the faces incident at e. Let

$$\vec{U}^+=\frac{\vec{U}_1+\vec{U}_2}{\|\vec{U}_1+\vec{U}_2\|}\text{ and }\vec{U}^-=\frac{\vec{U}_2-\vec{U}_1}{\|\vec{U}_2-\vec{U}_1\|}.$$

Choose coordinates (x,y,z) for $\mathbb{R}^3$ so that e lies along the x-axis, $\vec{U}^-$ lies on the positive y-axis and $\vec{U}^-$ lies on the positive z-axis. Define β to be the angle from $\vec{U}_1$ to $\vec{U}_2$ in the yz-plane.

If $e \subset \partial T$, let $\vec{T}$ be the tangent vector to e, $\vec{U}$ be the unit normal vector to the face containing e, and $\vec{V} = \vec{U} \times \vec{T}$ so that $(\vec{T}, \vec{V}, \vec{U})$ is an positively oriented frame.

Theorem 8. *Let $T \subseteq \mathbb{R}^3$ be an oriented triangulated surface. For any Borel set $B \subseteq \mathbb{R}^3$,*

$$\begin{aligned}
\phi_T^{H\vec{U}}(B) &= \sum_{e \subset \operatorname{int} T \text{ an edge}} -2\sin(\beta/2)\vec{U}^+ ds \wedge [e](B) + \sum_{e \subset \partial T \text{ an edge}} \vec{V} ds \wedge [e](B), \\
&= \sum_{e \subset \operatorname{int} T \text{ an edge}} -2\sin(\beta/2)\operatorname{length}(e \cap B)\vec{U}^+ + \sum_{e \subset \partial T \text{ an edge}} \vec{V} \operatorname{length}(e \cap B).
\end{aligned}$$

Proof. The proof is similar to the computation of the mean curvature vector measure for a smooth surface. First we observe that the $\omega^{H\vec{U}}|_{\operatorname{spt} N(T)}$ is supported above the edges of T. We can prove this by considering the normal cycle above the faces, edges, and vertices of T in much that same we did for the proof of the Gauss curvature measure theorem for a triangulated surface.

Let $B \subseteq \mathbb{R}^3$ be a Borel set and let be an edge of T. If $e \subset \operatorname{int} T$, assume e is oriented so that $\vec{U}^- \times \vec{U}^+$ is the positive tangent vector to e. Let $i_e : e \times \mathbb{R} \to T\mathbb{R}^3$ be the map $i_e(p,\theta) = (p, \vec{U}^+ \cos\theta + \vec{U}^- \sin\theta)$. Then the normal cycle of T above e equals $i_{e\#}([e] \times [-\beta/2, \beta/2]) - i_{e\#}([e] \times [\pi - \beta/2, \pi + \beta/2])$, i.e. the normal cycle of T above e is a sum of two integral currents associated with cylindrical surfaces in $T\mathbb{R}^3$ with opposite orientations (see Figure 3). As we showed in the mean curvature vector theorem for a smooth surface, $\omega^{H\vec{U}}\left(\frac{di_e}{ds}, \frac{di_e}{d\theta}\right) = -n$. Integrating over both cylindrical parts of the normal cycle above e yields

$$\begin{aligned}
\langle N_e(T) \llcorner \pi^{-1}(B), \omega^{H\vec{U}} \rangle &= \int_{e\cap B} \int_{-\beta/2}^{\beta/2} -n d\theta ds - \int_{e \cap B} \int_{\pi-\beta/2}^{\pi+\beta/2} -n d\theta ds \\
&= -2 \int_{e\cap B} \int_{-\beta/2}^{\beta/2} (\vec{U}^+ \cos\theta + \vec{U}^- \sin\theta) d\theta ds \\
&= -2 \int_{e \cap B} 2\sin(\beta/2)\vec{U}^+ ds \\
&= -4\sin(\beta/2)\vec{U}^+ \operatorname{length}(e \cap B).
\end{aligned}$$

Similarly if $e \subset \partial T$, the normal cycle of T above e equals $i_{\partial\#}([e] \times [0,\pi])$ where $i_\partial(p,\theta) = (p, -\vec{U}(p)\cos\theta - \vec{V}(p)\sin\theta)$. Then, much like we had in the proof of the mean curvature vector measure theorem for a smooth surface,

$$\langle N_e(T) \llcorner \pi^{-1}(B), \omega^{H\vec{U}} \rangle = 2\vec{V} \wedge ds[\partial T](B).$$

The result follows by summing the mean curvature vector measure over all the edges of T. □

The formula in Theorem 8 was previously discussed by Sullivan [13], who *defined* the mean curvature vector measure on a triangulated surface to be the right-hand side of the equation in Theorem 8. By contrast, we define $\phi_T^{H\vec{U}}$ using the normal cycle and then derive the equation in Theorem 8. Sullivan's definition was motivated by a force balance equation for mean curvature in the smooth case, described in the next section.

5.3 Physical interpretation

One way to think of the mean curvature vector is as the first variation of area. Let M be a smooth surface. Consider a variational vector field $\vec{X}$ on M that is supported away from the boundary of M. The vector field then gives us a family of smooth surfaces $M_t = \{p + t\vec{X}(p) : p \in M\}$. Let $A(t)$ denote the area of M_t. Then (see [1])

$$A'(0) = -\int_M \vec{X} \bullet H\vec{U} dA.$$

With respect to the L^2 inner product $\langle \vec{X}, \vec{Y} \rangle = \int_M \vec{X} \bullet \vec{Y} dA$, we can regard $-H\vec{U}$ as the gradient of the area, called the first variation of area [13]. If we take our variation vector field $\vec{X}$ to be $H\vec{U}$, then $A'(0) \leq 0$. In other words, area initially decreases when we deform M in the direction $H\vec{U}$ [1, p. 201]. This fact is the basis for algorithms designed to smooth triangulated surfaces. Recall from multivariable calculus that a function is minimized when its gradient is zero. Similarly, $A'(0) = 0$ for all variational vectors fields $\vec{X}$ when $H\vec{U} = 0$, so the surface area of a surface with a fixed boundary is minimal when $H = 0$. Surfaces with zero mean curvature everywhere are called minimal surfaces [1, p. 197].

The mean curvature vector has an analogy to the Gauss Bonnet formula, known as the force balance equation:

$$\int\int_M H\vec{U} ds + \int_{\partial M} \vec{V} ds = 0.$$

This formula represents a balance between surface tension along ∂M in the $\vec{V}$ direction and pressure forces acting on and normal to M [13]. In terms of curvature measures, the force balance equation is equivalent to $\phi_M^{H\vec{U}}(M) = 0$. As with the Gauss Bonnet formula, there is a simple proof of this fact for triangulated surfaces.

Theorem 9. *For an oriented triangulated surface $T \subseteq \mathbb{R}^3$, $\phi_T^{H\vec{U}}(T) = 0$.*

Proof. We will first prove the result when T is a triangle. Without loss of generality, suppose T lies in the xy-plane. Going around the triangle clockwise, label the vertices A, B, and C. Let $\vec{a} = \vec{BC}$, $\vec{b} = \vec{CA}$, and $\vec{c} = \vec{AB}$. The unit normal to each side in the plane containing T is obtained by rotating the tangent vectors $\vec{a}, \vec{b}, \vec{c}$ 90-degrees clockwise, denoted by $R(\vec{a})$, etc. The mean curvature vector measure of side BC is $R(\vec{a})/|a| \cdot |a| = R(\vec{a})$ and similarly for the other sides. Hence

$$\phi_T^{H\vec{U}}(T) = R(\vec{a}) + R(\vec{b}) + R(\vec{c}) = R(\vec{a} + \vec{b} + \vec{c}) = 0.$$

Now let T be any triangulated surface. For an edge e of T, it follows from the proof of Theorem 8 that

$$\phi_e^{H\vec{U}}(T) = \langle N(e), \omega^{H\vec{U}} \rangle = \int_e \int_0^{2\pi} -n d\theta ds = \int_e \int_0^{2\pi} (\vec{U}^+ \cos\theta + \vec{U}^- \sin\theta) d\theta ds = 0.$$

Also $\phi_v^{H\vec{U}}(T) = 0$ for a vertex v of T since $\omega^{H\vec{U}}|_{\operatorname{spt} N(v)} = 0$. So by Corollary 1,

$$\phi_T^{H\vec{U}}(T) = \sum_{f \subset T \text{ a face}} \phi_f^{H\vec{U}}(T) - \sum_{e \subset \operatorname{int} T \text{ an edge}} \phi_e^{H\vec{U}}(T) + \sum_{v \in \operatorname{int} T \text{ a vertex}} \phi_v^{H\vec{U}}(T) = 0$$

as required. □

6 Second fundamental form

All information about the curvature of a surface is encoded in the second fundamental form. Recall the scalar second fundamental form h of a smooth surface M is the symmetric bilinear form on vectors defined so that at each point $p \in M$, $h_p(\vec{X}, \vec{Y}) = -\nabla_{\vec{X}} \vec{U} \bullet \vec{Y}$ for vectors $\vec{X}, \vec{Y} \in T_pM$, where $\vec{U}$ is the unit normal vector to M. The eigenvalues κ_1, κ_2 of the linear operator associated with h_p are called the principal curvatures and the corresponding eigenvectors $\vec{e}_1, \vec{e}_2$ are called the principal directions. An equivalent way to define the scalar second fundamental form is by $h_p(\vec{X}, \vec{Y}) = \nabla_{\vec{X}} \vec{Y} \bullet \vec{U}$ for vectors $\vec{X}, \vec{Y} \in T_pM$. We define the vector second fundamental form $I\!I$ by $I\!I_p(\vec{X}, \vec{Y}) = (\nabla_{\vec{X}} \vec{Y})^{\perp}$ for $\vec{X}, \vec{Y} \in T_pM$, where $(\vec{Z})^{\perp}$ denotes the projection of the vector $\vec{Z}$ onto the span of $\vec{U}$. The scalar and vector second fundamental form are related by $I\!I(\vec{X}, \vec{Y}) = h(\vec{X}, \vec{Y})\vec{U}$.

We want to derive a curvature measure $\phi_M^{I\!I(\vec{X}, \vec{Y})}$ for smooth and triangulated surfaces with boundary such that in the special case that M is a smooth surface without boundary,

$$\phi_M^{I\!I(\vec{X}, \vec{Y})}(B) = \int_{M \cap B} I\!I(\vec{X}, \vec{Y}) dA.$$

Such a measure can be used to compute the principal curvature and directions for triangulated surfaces. We study the measure that corresponds to $I\!I$ rather than to h since $I\!I$ is independent of the orientation of M whereas h depends on the orientation of M. We shall combine the approach from [10] and our approach to studying the mean curvature vector.

Recall $h = \kappa_1 \vec{e}_1 \otimes \vec{e}_1 + \kappa_2 \vec{e}_2 \otimes \vec{e}_2$. Define $\widetilde{h}$ to be the bilinear form defined by $\widetilde{h} = \kappa_2 \vec{e}_1 \otimes \vec{e}_1 + \kappa_1 \vec{e}_2 \otimes \vec{e}_2$. Define the 2-forms $\omega^{h(\vec{X}, \vec{Y})}$ and $\omega^{\widetilde{h}(\vec{X}, \vec{Y})}$ so that at $(p, n) \in T\mathbb{R}^3$,

$$\begin{aligned} \omega_{(p,n)}^{h(\vec{X}, \vec{Y})} &= (n \times \vec{X}, 0) \wedge (\vec{Y}, 0), \\ \omega_{(p,n)}^{\widetilde{h}(\vec{X}, \vec{Y})} &= (\vec{X}, 0) \wedge (n \times \vec{Y}, 0). \end{aligned}$$

In [10] the measures $\phi_M^{h(\vec{X}, \vec{Y})}$ and $\phi_M^{\widetilde{h}(\vec{X}, \vec{Y})}$ are defined for a smooth surface M without boundary by

$$\begin{aligned} \phi_M^{h(\vec{X}, \vec{Y})}(B) &= -\langle i_{+\#}[M] \llcorner \pi^{-1}(B), \omega^{h(\vec{X}, \vec{Y})} \rangle, \\ \phi_M^{\widetilde{h}(\vec{X}, \vec{Y})}(B) &= -\langle i_{+\#}[M] \llcorner \pi^{-1}(B), \omega^{\widetilde{h}(\vec{X}, \vec{Y})} \rangle, \end{aligned}$$

so that

$$\phi_M^{h(\vec{X}, \vec{Y})}(B) = \int_{M \cap B} h(\vec{X}, \vec{Y}) dA$$
$$\phi_M^{\widetilde{h}(\vec{X}, \vec{Y})}(B) = \int_{M \cap B} \widetilde{h}(\vec{X}, \vec{Y}) dA.$$

(Note that [10] defines h using the sign convention $h(\vec{X}, \vec{Y}) = \nabla_{\vec{X}} \vec{U} \bullet \vec{Y}$.) Define $\widetilde{I\!I}$ by $\widetilde{I\!I}(\vec{X}, \vec{Y}) = \widetilde{h}(\vec{X}, \vec{Y})\vec{U}$ for $\vec{X}, \vec{Y} \in T_pM$.

Definition. Let $M \subseteq \mathbb{R}^3$ be a smooth or triangulated surface. Define the vector-valued 2-forms $\omega^{I\!I(\vec{X}, \vec{Y})}$ and $\omega^{\widetilde{I\!I}(\vec{X}, \vec{Y})}$ by $\omega_{(p,n)}^{I\!I(\vec{X}, \vec{Y})} = -n\omega_{(p,n)}^{h(\vec{X}, \vec{Y})}$ and $\omega_{(p,n)}^{\widetilde{I\!I}(\vec{X}, \vec{Y})} = -n\omega_{(p,n)}^{\widetilde{h}(\vec{X}, \vec{Y})}$ for $p \in \mathbb{R}^3$, $n \in T_p\mathbb{R}^3$, and vectors $\vec{X}, \vec{Y}$. Define the vector-valued measures $\phi_M^{I\!I(\vec{X}, \vec{Y})}$ and $\phi_M^{\widetilde{I\!I}(\vec{X}, \vec{Y})}$ by

$$\begin{aligned} \phi_M^{I\!I(\vec{X}, \vec{Y})}(B) &= \tfrac{1}{2} \langle N(M) \llcorner \pi^{-1}(B), \omega_{(p,n)}^{I\!I(\vec{X}, \vec{Y})} \rangle, \\ \phi_M^{\widetilde{I\!I}(\vec{X}, \vec{Y})}(B) &= \tfrac{1}{2} \langle N(M) \llcorner \pi^{-1}(B), \omega_{(p,n)}^{\widetilde{I\!I}(\vec{X}, \vec{Y})} \rangle, \end{aligned}$$

for all Borel sets $B \subseteq \mathbb{R}^3$. Define the vector-valued bilinear forms $\phi_M^{I\!I}(B)$ and $\phi_M^{\widetilde{I\!I}}(B)$ by $\left(\phi_M^{I\!I}(B)\right)(\vec{X},\vec{Y}) = \phi_M^{I\!I(\vec{X},\vec{Y})}(B)$ and $\left(\phi_M^{\widetilde{I\!I}}(B)\right)(\vec{X},\vec{Y}) = \phi_M^{\widetilde{I\!I}(\vec{X},\vec{Y})}(B)$. Note that $\phi_M^{I\!I}$ and $\phi_M^{\widetilde{I\!I}}$ are measures taking values in the vector space of vector-valued bilinear forms.

6.1 Smooth surfaces with boundary

Theorem 10. *Let $M \subseteq \mathbb{R}^3$ be an oriented smooth surface with a smooth boundary and let $(\vec{T},\vec{V},\vec{U})$ be a Darboux frame on the boundary of M. Then for any Borel set $B \subseteq \mathbb{R}^3$,*

$$\begin{aligned}\phi_M^{I\!I} &= I\!I dA[M] + \tfrac{1}{3}\{-(\vec{U}\otimes\vec{V}+\vec{V}\otimes\vec{U})\vec{U} + (2\vec{U}\otimes\vec{U}+\vec{V}\otimes\vec{V})\vec{V}\}ds \wedge [\partial M],\\ \phi_M^{\widetilde{I\!I}} &= \widetilde{I\!I} dA[M] + (\vec{T}\otimes\vec{T})\vec{V} ds \wedge [\partial M].\end{aligned}$$

Proof. Let $B \subseteq \mathbb{R}^3$ be a Borel set. Recall $N(M) = [i_+(M)] - [i_-(M)] + [i_\partial(\partial M \times [0,\pi])]$. Recall $\vec{e}_1, \vec{e}_2$ are the principal directions on M. Then

$$\begin{aligned}\omega^{I\!I(\vec{X},\vec{Y})}_{(p,\vec{U}(p))}(\nabla_{\vec{e}_1} i_+, \nabla_{\vec{e}_2} i_+) &= -\vec{U}(p)\begin{vmatrix} (\vec{X}\times\vec{U})\bullet\vec{e}_1 & \vec{Y}\bullet\nabla_{\vec{e}_1}\vec{U} \\ (\vec{X}\times\vec{U})\bullet\vec{e}_2 & \vec{Y}\bullet\nabla_{\vec{e}_2}\vec{U}\end{vmatrix}\\ &= -\vec{U}(p)\begin{vmatrix} \vec{X}\bullet(\vec{U}\times\vec{e}_1) & \vec{Y}\bullet\nabla_{\vec{e}_1}\vec{U} \\ \vec{X}\bullet(\vec{U}\times\vec{e}_2) & \vec{Y}\bullet\nabla_{\vec{e}_2}\vec{U}\end{vmatrix}\\ &= -\vec{U}(p)\begin{vmatrix} \vec{X}\bullet\vec{e}_2 & \vec{Y}\bullet-\kappa_1\vec{e}_1 \\ \vec{X}\bullet-\vec{e}_1 & \vec{Y}\bullet-\kappa_2\vec{e}_2\end{vmatrix}\\ &= (\kappa_1(\vec{e}_1\bullet\vec{X})(\vec{e}_1\bullet\vec{Y}) + \kappa_2(\vec{e}_2\bullet\vec{X})(\vec{e}_2\bullet\vec{Y}))\vec{U}(p)\\ &= (\kappa_1\vec{e}_1\otimes\vec{e}_1 + \kappa_2\vec{e}_2\otimes\vec{e}_2)(\vec{X},\vec{Y})\vec{U}(p) = I\!I(\vec{X},\vec{Y}).\end{aligned}$$

Similarly $\omega^{I\!I(\vec{X},\vec{Y})}_{(p,\vec{U}(p))}(\nabla_{\vec{e}_1} i_-, \nabla_{\vec{e}_2} i_-) = -I\!I(\vec{X},\vec{Y})$. So

$$\langle (i_{+\#}[M] - i_{-\#}[M])\llcorner\pi^{-1}(B), \omega^{I\!I(\vec{X},\vec{Y})}\rangle = \int_M 2I\!I(\vec{X},\vec{Y})dA = 2I\!I(\vec{X},\vec{Y})dA \wedge [M](B). \quad (14)$$

Now observe $(\vec{u}_1,\vec{u}_2,n) = (\vec{T}, \vec{U}\sin\theta - \vec{V}\cos\theta, -\vec{U}\cos\theta - \vec{V}\sin\theta)$ is an orthonormal frame on the boundary of M. Recall from multivariable calculus that

$$\langle i_{\partial\#}([\partial M]\times[0,\pi])\llcorner\pi^{-1}(B), \omega^{I\!I(\vec{X},\vec{Y})}\rangle = \int_{\partial M\cap B}\int_0^\pi \omega^{I\!I(\vec{X},\vec{Y})}\left(\tfrac{di_\partial}{ds}, \tfrac{di_\partial}{d\theta}\right) d\theta ds$$

where we think of $p(s)$ as a unit speed parameterization of the boundary of M. Since $\frac{dn}{d\theta} = \vec{u}_2$,

$$\begin{aligned}\omega^{I\!I(X,Y)}\left(\tfrac{di_\partial}{d\theta}, \tfrac{di_\partial}{ds}\right) &= -n\begin{vmatrix} (\vec{X}\times n)\bullet\vec{u}_1 & \vec{Y}\bullet\frac{dn}{ds} \\ (\vec{X}\times n)\bullet 0 & \vec{Y}\bullet\frac{dn}{d\theta}\end{vmatrix} = -n\begin{vmatrix} \vec{X}\bullet(n\times\vec{u}_1) & \vec{Y}\bullet\frac{dn}{ds} \\ 0 & \vec{Y}\bullet\frac{dn}{d\theta}\end{vmatrix}\\ &= -n\begin{vmatrix} \vec{X}\bullet\vec{u}_2 & \vec{Y}\bullet\frac{dn}{ds} \\ 0 & \vec{Y}\bullet\vec{u}_2\end{vmatrix} = -n(u_2\otimes u_2)(\vec{X},\vec{Y}).\end{aligned}$$

Hence

$$
\begin{aligned}
&\int_{i_\partial(\partial M\cap B\times[0,\pi])}\omega^{\mathit{I\!I}(\vec{X},\vec{Y})}\\
=&\int_{\partial M\cap B}\int_0^\pi -n(\vec{u}_2\otimes\vec{u}_2)(\vec{X},\vec{Y})d\theta ds\\
=&\int_{\partial M\cap B}\int_0^\pi((\vec{U}\otimes\vec{U}\cos\theta\sin^2\theta-(\vec{V}\otimes\vec{U}+\vec{U}\otimes\vec{V})\cos^2\theta\sin\theta+\vec{V}\otimes\vec{V}\cos^3\theta)(\vec{X},\vec{Y})\vec{U}\\
&+(\vec{U}\otimes\vec{U}\sin^3\theta-(\vec{V}\otimes\vec{U}+\vec{U}\otimes\vec{V})\cos\theta\sin^2\theta+\vec{V}\otimes\vec{V}\cos^2\theta\sin\theta)(\vec{X},\vec{Y})\vec{V})d\theta ds\\
=&\int_{\partial M\cap B}\tfrac{2}{3}(-(\vec{U}\otimes\vec{V}+\vec{V}\otimes\vec{U})(\vec{X},\vec{Y})\vec{U}+(2\vec{U}\otimes\vec{U}+\vec{V}\otimes\vec{V})(\vec{X},\vec{Y})\vec{V})ds\\
=&\tfrac{2}{3}\{-(\vec{U}\otimes\vec{V}+\vec{V}\otimes\vec{U})(\vec{X},\vec{Y})\vec{U}+(2\vec{U}\otimes\vec{U}+\vec{V}\otimes\vec{V})(\vec{X},\vec{Y})\vec{V}\}ds[\partial M](B). \qquad (15)
\end{aligned}
$$

The formula for $\phi_M^{\mathit{I\!I}}$ follows from (14) and (15).

Similarly, $\omega^{\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})}_{(p,\vec{U}(p))}(\nabla_{\vec{e}_1}i_+,\nabla_{\vec{e}_2}i_+)=\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})$, $\omega^{\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})}_{(p,\vec{U}(p))}(\nabla_{\vec{e}_1}i_-,\nabla_{\vec{e}_2}i_-)=-\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})$, and

$$
\omega^{\widetilde{\mathit{I\!I}}(X,Y)}\left(\tfrac{di_\partial}{d\theta},\tfrac{di_\partial}{ds}\right)=-n\begin{vmatrix}(\vec{X}\times n)\bullet\frac{dn}{ds} & \vec{Y}\bullet\vec{u}_1\\ (\vec{X}\times n)\bullet\frac{dn}{d\theta} & \vec{Y}\bullet 0\end{vmatrix}=-n(\vec{u}_1\otimes\vec{u}_1)(\vec{X},\vec{Y}),
$$

so

$$
\langle(i_{+\#}[M]-i_{-\#}[M])\llcorner\pi^{-1}(B),\omega^{\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})}\rangle=\int_M 2\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})dA=2\widetilde{\mathit{I\!I}}(\vec{X},\vec{Y})dA\wedge[M](B).
$$

and

$$
\begin{aligned}
\int_{i_\partial(\partial M\cap B\times[0,\pi])}\omega^{\widetilde{\mathit{I\!I}}(X,Y)} &= \int_{\partial M\cap B}\int_0^\pi -n(\vec{u}_1\otimes\vec{u}_1)(\vec{X},\vec{Y})d\theta ds\\
&= \int_{\partial M\cap B}\int_0^\pi(\vec{V}\cos\theta+\vec{U}\sin\theta)(\vec{T}\otimes\vec{T})(\vec{X},\vec{Y})d\theta ds\\
&= \int_{\partial M\cap B}2(\vec{T}\otimes\vec{T})(\vec{X},\vec{Y})\vec{U}ds\\
&= 2(\vec{T}\otimes\vec{T})(\vec{X},\vec{Y})\vec{V}ds\wedge[\partial M](B).
\end{aligned}
$$

The formula for $\phi_M^{\widetilde{\mathit{I\!I}}}$ follows. □

6.2 Triangulated surfaces with boundary

Theorem 11. *Let $T\subseteq\mathbb{R}^3$ be an oriented triangulated surface. For any Borel set $B\subseteq\mathbb{R}^3$,*

$$
\begin{aligned}
\phi_T^{\mathit{I\!I}} &= \sum_{e\subset\operatorname{int}T\ an\ edge}\tfrac{2}{3}(-(\sin^3\tfrac{\beta}{2}\vec{U}^+\otimes\vec{U}^+-(2\sin\tfrac{\beta}{2}+\sin^3\tfrac{\beta}{2})\vec{U}^-\otimes\vec{U}^-)\vec{U}^+\\
&\quad-\sin^3\tfrac{\beta}{2}(\vec{U}^+\otimes\vec{U}^-+\vec{U}^-\otimes\vec{U}^+)\vec{U}^-)ds\wedge[e](B)\\
&\quad+\sum_{e\subset\partial T\ an\ edge}\tfrac{1}{3}(-(\vec{U}\otimes\vec{V}+\vec{V}\otimes\vec{U})\vec{U}+(2\vec{U}\otimes\vec{U}+\vec{V}\otimes\vec{V})\vec{V})ds\wedge[e],\\
\phi_T^{\widetilde{\mathit{I\!I}}} &= \sum_{e\subset\operatorname{int}T\ an\ edge}2\sin\tfrac{\beta}{2}(\vec{T}\otimes\vec{T})\vec{U}^+ds\wedge[e]-\sum_{e\subset\partial T\ an\ edge}(\vec{T}\otimes\vec{T})\vec{V}ds\wedge[e].
\end{aligned}
$$

Proof. The proof is similar to the computation of the second fundamental form vector measures on the boundary of a smooth surface. Let $B \subseteq \mathbb{R}^3$ be a Borel set. First we observe that the $\omega^{I\!I(\vec{X},\vec{Y})}|_{\operatorname{spt} N(T)}$ is supported above the edges of T. Let e be an edge of T. Recall from the proof of formula for $\phi_T^{H\vec{U}}$, the normal cycle of T above e equals $i_{e\#}([e]\times[-\beta/2,\beta/2]) - i_{e\#}([e]\times[\pi-\beta/2,\pi+\beta/2])$. As we showed in the second fundamental form vector theorem for a smooth surface, $i_e^*\omega^{I\!I(\vec{X},\vec{Y})} = n(\vec{u}_2\otimes\vec{u}_2)(\vec{X},\vec{Y})d\theta ds$ and $i_e^*\omega^{\widetilde{I\!I}(\vec{X},\vec{Y})} = n(\vec{u}_1\otimes\vec{u}_1)(\vec{X},\vec{Y})d\theta ds$, where $(\vec{u}_1,\vec{u}_2,n) = (\vec{T}, -\vec{U}^+\sin\theta + \vec{U}^-\cos\theta, \vec{U}^+\cos\theta + \vec{U}^-\sin\theta)$. Integrating over both cylindrical parts of the normal cycle above e,

$$\begin{aligned}
&\langle N(M)\llcorner\pi^{-1}(e\cap B), \omega^{I\!I(X,Y)}\rangle\\
=\ & 2\int_{e\cap B}\int_{-\beta/2}^{\beta/2} -n(\vec{u}_2\otimes\vec{u}_2)(\vec{X},\vec{Y})d\theta ds\\
=\ & -2\int_{e\cap B}\int_{-\beta/2}^{\beta/2}((\vec{U}^+\otimes\vec{U}^+\cos\theta\sin^2\theta - (\vec{U}^+\otimes\vec{U}^- + \vec{U}^-\otimes\vec{U}^+)\cos^2\theta\sin\theta\\
&+\vec{U}^-\otimes\vec{U}^-\cos^3\theta)\vec{U}^+ + ((\vec{U}^+\otimes\vec{U}^+\sin^3\theta - (\vec{U}^+\otimes\vec{U}^- + \vec{U}^-\otimes\vec{U}^+)\cos\theta\sin^2\theta\\
&+\vec{U}^-\otimes\vec{U}^-\cos^2\theta\sin\theta)\vec{U}^-)(\vec{X},\vec{Y})d\theta ds\\
=\ & \tfrac{-4}{3}\int_{e\cap B}((\sin^3\tfrac{\beta}{2}\vec{U}^+\otimes\vec{U}^+ + (2\sin\tfrac{\beta}{2} - \sin^3\tfrac{\beta}{2})\vec{U}^-\otimes\vec{U}^-)\vec{U}^+\\
&-\sin^3\tfrac{\beta}{2}(\vec{U}^+\otimes\vec{U}^- + \vec{U}^-\otimes\vec{U}^+)\vec{U}^-)(\vec{X},\vec{Y})ds\\
=\ & \tfrac{-4}{3}((\sin^3\tfrac{\beta}{2}\vec{U}^+\otimes\vec{U}^+ + (2\sin\tfrac{\beta}{2} - \sin^3\tfrac{\beta}{2})\vec{U}^-\otimes\vec{U}^-)\vec{U}^+\\
&-\sin^3\tfrac{\beta}{2}(\vec{U}^+\otimes\vec{U}^- + \vec{U}^-\otimes\vec{U}^+)\vec{U}^-)(\vec{X},\vec{Y})ds\wedge[e](B).
\end{aligned}$$

and

$$\begin{aligned}
\langle N(M)\llcorner\pi^{-1}(e\cap B), \omega^{\widetilde{I\!I}(\vec{X},\vec{Y})}\rangle &= 2\int_{e\cap B}\int_{-\beta/2}^{\beta/2} n(\vec{u}_1\otimes\vec{u}_1)(\vec{X},\vec{Y})d\theta ds\\
&= 2\int_{e\cap B}\int_{-\beta/2}^{\beta/2}(U^+\cos\theta + U^-\sin\theta)(\vec{T}\otimes\vec{T})(\vec{X},\vec{Y})d\theta ds\\
&= 4\int_{e\cap B} U^+\sin\tfrac{\beta}{2}(\vec{T}\otimes\vec{T})(\vec{X},\vec{Y})ds\\
&= 4U^+\sin\tfrac{\beta}{2}(\vec{T}\otimes\vec{T})(\vec{X},\vec{Y})ds\wedge[e](B).
\end{aligned}$$

The computation for boundary edges is the same as for smooth surfaces. The result follows by summing all the edges of M. □

7 Convergence Theorem

We wish to establish a theorem showing that the curvature measures of a triangulated surface approximate the curvature measures of a smooth surface when the triangulated surface is itself a good approximation of the smooth surface. We take the approach used by Cohen-Steiner and Morvan, using geometric measure theory to first establish a bound on the flat norm of the difference between the normal cycles of a smooth surface and an approximating triangulated surface [9].

7.1 Analysis of currents

To describe and prove our convergence theorem, we first need to define some norms on currents. Since currents are linear functionals on differential forms, which are themselves linear functionals on tangent vectors, the natural way to define the norms on currents is by first analyzing norms on tangent vectors and the induced norm on differential forms.

For a point $p \in \mathbb{R}^3$ and a vector $\vec{v} \in T_p\mathbb{R}^3$, we define $\|\vec{v}\|$ to be the standard Euclidean norm of $\vec{v}$. Recall $T\mathbb{R}^3 \simeq \mathbb{R}^3 \times \mathbb{R}^3$, so any point in $T\mathbb{R}^3$ may be represented as (p, n) for $p, n \in \mathbb{R}^3$. Furthermore, any vector in $T_{(p,n)}(T\mathbb{R}^3)$ can be written as $(\vec{v}, \vec{w})$ for $\vec{v} \in T_p\mathbb{R}^3$ and $\vec{w} \in T_n\mathbb{R}^3$. We define $\|(\vec{v}, \vec{w})\|_{T\mathbb{R}^3} = \max\{\|\vec{v}\|, \|\vec{w}\|\}$.

Given a 1-form $\omega_{(p,n)}$ at a point $(p, n) \in T\mathbb{R}^3$, we define [2, 1.8.1]

$$\|\omega_{(p,n)}\| = \sup\{\omega_{(p,n)}(\vec{v}) : \|\vec{v}\|_{T\mathbb{R}^3} \leq 1\}.$$

Given a 2-form $\omega_{(p,n)}$ at a point $(p, n) \in T\mathbb{R}^3$, we define

$$\|\omega_{(p,n)}\| = \sup\{\omega_{(p,n)}(\vec{v}_1, \vec{v}_2) : \|\vec{v}_1\|_{T\mathbb{R}^3} \leq 1, \|\vec{v}_2\|_{T\mathbb{R}^3} \leq 1\}.$$

For a differential form ω on $T\mathbb{R}^3$, we define $\|\omega\| = \sup_{(p,n)\in T\mathbb{R}^3} \|\omega_{(p,n)}\|$.

Let $f : \mathcal{U} \to \mathbb{R}^m$, where $\mathcal{U} \subseteq \mathbb{R}^n$ is open, and let Df denote the first derivative of f, provided it is defined. Let

$$Df(\vec{v}) = \lim_{h\to 0} \frac{f(p + h\vec{v}) - f(p)}{h}$$

be the derivative of f in the direction $\vec{v}$. We define [2, 3.1.1]

$$\|Df\| = \sup\{\|Df(\vec{v})\| : \|\vec{v}\| \leq 1\}.$$

We can represent Df as an $m \times n$ matrix, which we will also denote by Df, and thus compute its norm using linear algebra. In the case that $\mathbb{R}^m$ and $\mathbb{R}^n$ both have the standard Euclidean metric, $\|Df\|$ equals the square root of the largest eigenvalue of $(Df)^T Df$ [6, p. 281]. Furthermore, if $m = n$ and Df is a normal matrix at each point in $\mathcal{U}$, $\|Df\|$ equals the largest absolute value of the eigenvalues of Df. If we replace $\mathbb{R}^m$ with $T\mathbb{R}^3 \simeq \mathbb{R}^6$ and let $n = 3$, we can represent Df as a 2×1 block matrix with 3×3 blocks. The norm of Df is then computed as the maximum norm of the two blocks.

Given an current S, we define the *mass norm* $\mathbf{M}[S]$ of S by [2, 4.1.7]

$$\mathbf{M}[S] = \sup\{S(\omega) : \|\omega\| \leq 1\},$$

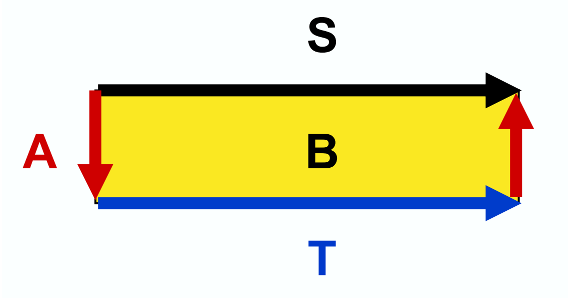
and we define the *flat norm* $\mathbf{F}[S]$ of S by [2, 4.1.12]

$$\begin{aligned} \mathbf{F}[S] &= \sup\{S(\omega) : \|\omega\| \leq 1, \|d\omega\| \leq 1\} \\ &= \inf\{\mathbf{M}[A] + \mathbf{M}[B] : S = A + \partial B,\ A, B \text{ are currents}\}. \end{aligned}$$

Geometrically, the flat norm gives a good indication for when two surfaces are close together. If M_1, M_2 are closed, compact, oriented smooth surfaces, then $\mathbf{F}([M_1] - [M_2])$ is small when the Hausdorff distance between M_1 and M_2 is small (see Figure 4). For this reason we will use the flat norm to obtain bounds on the difference of the normal cycles of a triangulated surface and a smooth surface.

The mass and flat norms are not norms on the entire space of currents on $T\mathbb{R}^3$ since the mass and flat norms of some currents are infinite. Thus we must restrict out attention to

Figure 4: The flat norm of $S - T$ gives a good indication of when the line segments associated with S and T are close together. Suppose that the lengths of the line segments are 1 and the distance between the line segments is ϵ, then $\mathbf{F}[S - T] = 3\epsilon$, whereas $\mathbf{M}[S - T] = 2$.



spaces of currents where the mass and flat norms are true norms. We define the space of normal currents to be the set of all currents S such that $\mathbf{M}[S] + \mathbf{M}[\partial S] < \infty$, [2, 4.1.7]. Examples of m-dimensional normal currents include finite sums of currents associated with closed, compact, oriented m-dimensional manifolds. We define the space of m-dimensional flat currents to be the flat norm closure of the space of m-dimensional normal currents in the space of all currents with compact support, [2, 4.1.12].

7.2 Constancy Theorem

To prove the convergence theorem we will also need Federer's Constancy Theorem. The Constancy Theorem can be stated in two equivalent ways (see [2, 4.1.31] and [8, Theorem 4.9]).

Theorem 12. *(Constancy Theorem) Let S be an m-dimensional flat current supported on an open set $\mathcal{U}$ with compact closure and $A \subset \mathcal{U}$ be a connected m-dimensional smooth manifold.*

1. *If $\operatorname{spt} S \setminus \operatorname{int} A$ is closed relative to $\mathcal{U}$ and $\operatorname{spt} \partial S \subseteq \mathcal{U} \setminus \operatorname{int} A$, then there exists a $c \in \mathbb{R}$ such that*
$$\operatorname{spt}(S - c[A]) \subseteq \mathcal{U} \setminus \operatorname{int} A.$$
Furthermore, if S is an integral current, then c is an integer.

2. *If $\operatorname{spt} S \subseteq A$ and $\operatorname{spt} \partial S \subseteq \partial A$, then $S = c[A]$ for some $c \in \mathbb{R}$. Furthermore, if S is an integral current, then c is an integer.*

Corollary 2. *Let S be an m-dimensional flat current supported on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$ and $A \subset \mathbb{S}^2$ be a connected smooth surface. If $\operatorname{spt} S \setminus A$ is closed relative to $\mathbb{S}^2$ and $\operatorname{spt} \partial S \subseteq \mathbb{S}^2 \setminus A$, then*
$$\operatorname{spt}(S - c[A]) \subseteq \mathbb{S}^2 \setminus A,$$
for some $c \in \mathbb{R}$. Furthermore, if S is an integral current, then c is an integer.

Proof. Let $\mathcal{U}$ be an open neighborhood of $\mathbb{S}^2$. By the Constancy Theorem, there is a $c \in \mathbb{R}$ such that
$$\operatorname{spt}(S - c[A]) \subseteq \mathcal{U} \setminus A.$$
Note c is an integer if S is an integral current. Since $\operatorname{spt}(S - c[A]) \subseteq \mathbb{S}^2$, the claim follows. □

7.3 Statement of the theorem

We begin by stating a general bound on the difference of two curvature measures.

Definition. Let M be a smooth surface and let $A \subseteq \mathbb{R}^3$. We define the norm $\|\cdot\|_A$ on continuous functions $f : A \to \mathbb{R}$ by $\|f\|_A = 0$ if $A = \emptyset$ and $\|f\|_A = \sup\{|f(p)| : p \in A\}$ otherwise. Recall that h is the scalar second fundamental form of M and define $\|h\|_A$ by $\|h\|_A = 0$ if $A = \emptyset$ and

$$\|h\|_A = \sup\{|h_p(\vec{X}, \vec{Y})| : p \in A, \|\vec{X}\| = \|\vec{Y}\| = 1\}$$

otherwise. Let $d(x, A) = \inf_{y \in A} \|x - y\|$. Define σ_A by $\sigma_A = 1$ if $A \neq \emptyset$ and $\sigma_A = 0$ if $A = \emptyset$.

Let T be a triangulated surface homeomorphic to M. Let $\mathcal{U} \subseteq \mathbb{R}^3$ be an open set containing T and suppose that $\psi : \mathcal{U} \to M$ is a Lipschitz map such that $\psi|_T : T \to M$ is a homeomorphism. Let $B \subseteq \mathbb{R}^3$ be a nonempty Borel set contained in T. We say B is *regular* if $N(T) \llcorner \pi^{-1}(B)$ is an integral current [9]. Let $\delta_B = \sup_{x \in T \cap B} \|x - \psi(x)\|$. Define $\alpha_{\vec{U},B}$ to be the supremum over $x \in T \cap B$ of the angles between unit normal vectors to those faces whose closure contains x and the unit normal vector to M at $\psi(x)$. Define $\alpha_{\vec{T},B}$ by $\alpha_{\vec{T},B} = 0$ if $\partial T \cap B = \emptyset$ and otherwise $\alpha_{\vec{T},B}$ is the supremum over $x \in \partial T \cap B$ of the angles between edges whose closure contains x and the tangent vector to ∂M at $\psi(x)$.

Theorem 13. *Let M be a smooth surface and T be a triangulated surface homeomorphic to M. Let B be a nonempty regular Borel set contained in T. Let $\mathcal{U} \subseteq \mathbb{R}^3$ be an open set containing T, and let $\psi : \mathcal{U} \to M$ be a Lipschitz map with Lipschitz constant λ. Suppose that*

(1) $\psi|_T : T \to M$ is a homeomorphism,

(2) $\alpha_{\vec{U},B} < \pi/2$ and $\alpha_{\vec{T},B} < \pi/2$, and

(3) for almost all points x in a neighborhood of ∂T in T, $\vec{v} \bullet D\psi(\vec{v}) \geq 0$ for all vectors $\vec{v} \in T_x\mathbb{R}^3$.

Let ϕ be either the Gauss curvature measure or mean curvature normal measure. Then

$$\begin{aligned} |\phi_T(B) - \phi_M(\psi(B))| \leq\ & \tfrac{1}{2} C(\phi) \sup\{\delta_B, \alpha_{\vec{U},B} + \alpha_{\vec{T},B}\} \Big(\max\{1, \lambda\} \max\Big\{1, \|h\|_{M \cap \psi(B)}, \\ & 2\sqrt{\|\kappa_g\|^2_{\partial M \cap \psi(B)} + 2\|h\|^2_{\partial M \cap \psi(B)}} + \frac{\pi \sigma_{\partial M \cap \psi(B)}}{\pi - 2\alpha_{\vec{U},B}}\Big\}\Big)^2 \\ & \left(\mathbf{M}[N(T) \llcorner \pi^{-1}(B)] + \mathbf{M}[\partial(N(T) \llcorner \pi^{-1}(B))]\right), \end{aligned}$$

where κ_g is the geodesic curvature of ∂M. Furthermore,

$$\begin{aligned} C(\phi^G) &= \max\{\|\omega^G\|, \|d\omega^G\|\} = 3, \\ C(\phi^{H\vec{U}}) &= \sqrt{3} \max\{\|n_1 \omega^H\|, \|d(n_1 \omega^H)\|\} \leq 12\sqrt{3}. \end{aligned}$$

In the case that $\partial M = \emptyset$, the above inequality becomes

$$\begin{aligned} |\phi_T(B) - \phi_M(\psi(B))| \leq\ & \tfrac{1}{2} C(\phi) \max\{\delta_B, \alpha_{\vec{U},B}\} \left(\max\{1, \lambda\} \max\left\{1, \|h\|_{M \cap \psi(B)}\right\}\right)^2 \\ & \cdot \left(\mathbf{M}[N(T) \llcorner \pi^{-1}(B)] + \mathbf{M}[\partial(N(T) \llcorner \pi^{-1}(B))]\right). \end{aligned}$$

This is equivalent to the bound given in [9].

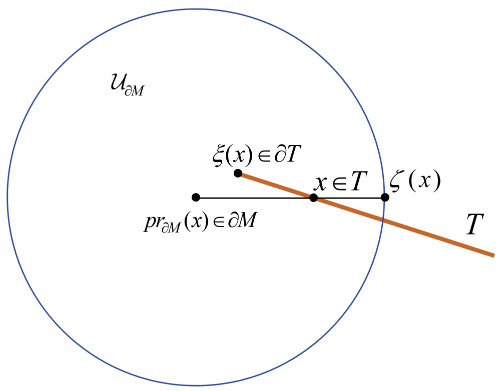
We want to describe the error bound in Theorem 13 in terms of an explicit choice for the function ψ. In [9], $\psi = \mathrm{pr}_M$, where $\mathrm{pr}_M(x)$ is the unique closest point on M to the point x.

However, in the case that M and T are surfaces with boundary, $\mathrm{pr}_M|_T$ is not necessarily a homeomorphism. Thus, we must use another map for ψ, which will be defined in terms of functions pr_M and $\mathrm{pr}_{\partial M}$ defined as follows. Given a point x, $\mathrm{pr}_M(x)$ is closest point on M to x and $\mathrm{pr}_{\partial M}(x)$ is the closest point on ∂M to x. To use such a construction we need the following lemma, which we will prove in Section 7.8.

Lemma 1. *Given a smooth surface M with boundary, there is an open set $\mathcal{U}_M \subseteq \mathbb{R}^3$ containing M on which we can define the map $\mathrm{pr}_M : \mathcal{U}_M \to M$ such that $\mathrm{pr}_M(x)$ is the unique closest point on M to $x \in \mathcal{U}_M$. Also, there is an open set $\mathcal{U}_{\partial M} \subseteq \mathbb{R}^3$ containing ∂M on which we can define the map $\mathrm{pr}_{\partial M} : \mathcal{U}_{\partial M} \to \partial M$ such that $\mathrm{pr}_{\partial M}(x)$ is the unique closest point on ∂M to $x \in \mathcal{U}_{\partial M}$.*

We can let $\mathcal{U}_M = \{x \in \mathbb{R}^3 : d(x, M) < r\}$; that is $\mathcal{U}_M$ is the tubular neighborhood of M of radius $r > 0$. It is well-known that such an r must be less $1/\|h\|_M$ [2]. Similarly, we can let $\mathcal{U}_{\partial M} = \{x \in \mathbb{R}^3 : d(x, \partial M) < r\}$ for $r > 0$ and such an r must be less than $1/\|\kappa\|_{\partial M}$, where κ is the curvature of ∂M. In what follows, suppose $\mathcal{U}_M = \{x \in \mathbb{R}^3 : d(x, M) < r\}$ and $\mathcal{U}_{\partial M} = \{x \in \mathbb{R}^3 : d(x, \partial M) < r\}$ for a fixed $r > 0$ and further suppose that $\partial T \subseteq \mathcal{U}_{\partial M}$.

Figure 5: The slice of $\mathcal{U}_{\partial M}$ and T in the plane perpendicular to ∂M at $\mathrm{pr}_{\partial M}(x)$. For $x \in T \cap \mathcal{U}_{\partial M}$, $\mathrm{pr}_{\partial M}(x)$ is the closest point to x on ∂M, $\xi(x)$ is the unique point on ∂T such that $\mathrm{pr}_{\partial M}(\xi(x)) = \mathrm{pr}_{\partial M}(x)$, and $\zeta(x)$ is the point on $\partial\mathcal{U}_{\partial M}$ that lies on the ray starting at $\mathrm{pr}_{\partial M}(x)$ and passing through x.



We will assume that $\mathrm{pr}_{\partial M}|_{\partial T} : \partial T \to \partial M$ is a homeomorphism. Then we may define $\xi : \mathcal{U}_{\partial M} \to \partial T$ so that $\xi(x) \in \partial T$ is the unique point such that $\mathrm{pr}_{\partial M}(\xi(x)) = \mathrm{pr}_{\partial M}(x)$ (see Figure 5). Define $\zeta : \mathcal{U}_{\partial M} \to \partial\mathcal{U}_{\partial M}$ by

$$\zeta(x) = \mathrm{pr}_{\partial M}(x) + \frac{r}{d(x, \partial M)}(x - \mathrm{pr}_{\partial M}(x))$$

for $x \in \mathcal{U}_{\partial M}$. Geometrically, $\zeta(x)$ is the point on $\partial\mathcal{U}_{\partial M}$ that lies on the ray starting at $\mathrm{pr}_{\partial M}(x)$ and passing through x (see Figure 5). Define the real-valued function w on points $x \in \mathcal{U}_{\partial M}$ such that $d(\xi(x), \partial M) < d(x, \partial M) < r$ by

$$w(x) = \frac{r - d(x, \partial M)}{r - d(\xi(x), \partial M)}$$

for $x \in \mathcal{U}_{\partial M}$. We regard w as a weight function with range that equals one on ∂T, equals zero on $\partial\mathcal{U}_{\partial M}$, and ranges from zero to one for $d(\xi(x), \partial M) < d(x, \partial M) < r$. We define

$b : \mathcal{U}_M \to \mathcal{U}_M$ by

$$b(x) = \begin{cases} \mathrm{pr}_{\partial M}(x) & \text{if } d(x, \partial M) < d(\xi(x), \partial M), \\ w(x)\,\mathrm{pr}_{\partial M}(x) + (1 - w(x))\zeta(x) & \text{if } d(\xi(x), \partial M) < d(x, \partial M) < r, \\ x & \text{if } d(x, \partial M) > r, \end{cases}$$

and we define $\psi = \mathrm{pr}_M \circ b$ so that $\psi : \mathcal{U} \to M$ where $\mathcal{U} = \mathcal{U}_M$. Conceptually, b stretches T so that $b(\partial T) = \partial M$ and then pr_M projects $b(T)$ onto M.

Theorem 14. *Let M, T, and B be defined as in Theorem 13. Let $\psi = \mathrm{pr}_M \circ b$. Assume the hypothesizes of Theorem 13 and further suppose that*

(a) the vertices of T lie on M and the vertices of ∂T lie on ∂M,

(b) $\mathrm{pr}_{\partial M}|_{\partial T}$ is a homeomorphism,

(c) $d(x, \partial M) < r$ for all $x \in \partial T$, where $r \leq \min\left\{\frac{1}{\|h\|_M}, \frac{1}{\|\kappa\|_{\partial M}}\right\}$.

Then Theorem 13 holds with

$$\begin{aligned} \delta_B &\leq 2\delta_{\partial M,B} + \delta_{M,B}, \\ \lambda &\leq \max\left\{\frac{1}{1 - r\|h\|_M}, \frac{1}{1 - r\|\kappa\|_{\partial M}}\right\} \\ &\quad \cdot \left(\frac{r}{r - \delta_{\partial M,\partial T}} \sec \alpha_{\vec{T},\partial T} + \max\left\{\frac{1}{1 - r\|\kappa\|_{\partial M}}, \frac{r}{r - \delta_{\partial M,\partial T}}\right\}\right). \end{aligned}$$

where $\delta_{\partial M,B} = \sup_{x \in \partial T \cap B} d(x, \partial M)$ and $\delta_{M,B} = \sup_{x \in T \cap B} d(x, M)$.

7.4 Overview of the proof

We wish to obtain a bound on $|\phi_T(B) - \phi_M(\psi(B))|$. Recall we define the curvature measure ϕ by $\phi_T(B) = \frac{1}{2}\langle N(T) \llcorner \pi^{-1}(B), \omega\rangle$ for any Borel set $B \subseteq \mathbb{R}^3$, where ω is the curvature 2-form corresponding to ϕ, and ϕ_M is defined similarly. Thus

$$\begin{aligned} \phi_T(B) - \phi_M(\psi(B)) &= \tfrac{1}{2}\langle N(T) \llcorner \pi^{-1}(B), \omega\rangle - \tfrac{1}{2}\langle N(M) \llcorner \pi^{-1}(\psi(B)), \omega\rangle \\ &= \tfrac{1}{2}\langle N(T) \llcorner \pi^{-1}(B) - N(M) \llcorner \pi^{-1}(\psi(B)), \omega\rangle. \end{aligned}$$

For simplicity, let $D = N(T) \llcorner \pi^{-1}(B)$ and $E = N(M) \llcorner \pi^{-1}(\psi(B))$. By the definition of the flat norm of a current,

$$|\phi^G_T(B) - \phi^G_M(\psi(B))| \leq \tfrac{1}{2}\mathbf{F}[D - E] \max\{\|\omega^G\|, \|d\omega^G\|\}.$$

We must be more careful with the mean curvature normal measure since it is a vector-valued measure. We can apply the flat norm definition to the real-valued components of $\phi^{H\vec{U}}_T(B) - \phi^{H\vec{U}}_M(\psi(B))$. Then $\|\phi^{H\vec{U}}_T(B) - \phi^{H\vec{U}}_M(\psi(B))\|^2$ is bounded above by the sum of the squares of $\frac{1}{2}\mathbf{F}[D - E] \max\{\|n_i\omega^H\|, \|d(n_i\omega^H)\|\}$, $i = 1, 2, 3$. By symmetry, $\max\{\|n_i\omega^H\|, \|d(n_i\omega^H)\|\}$ is the same for all i, so

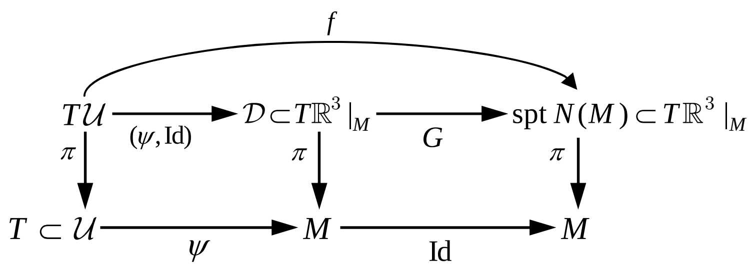
$$|\phi^{H\vec{U}}_T(B) - \phi^{H\vec{U}}_M(\psi(B))| \leq \tfrac{1}{2}\sqrt{3}\mathbf{F}[D - E] \max\{\|n_1\omega^H\|, \|d(n_1\omega^H)\|\}.$$

Now proving Theorem 13 reduces to computing the flat norm of $D - E$ and the norms of the curvature 2-forms and their exterior derivatives.

To obtain a bound on $\mathbf{F}[D-E]$, we shall use the homotopy formula for currents. Consider a homotopy $h : [0,1] \times T\mathcal{U} \to T\mathcal{U}$ defined by

$$h(t,p,n) = (1-t)(p,n) + tf(p,n)$$

for some function $f : T\mathcal{U} \to T\mathbb{R}^3$. We will define f to be a lift of ψ with the property that $f_\# D = E$. In [9], $f = \mathcal{G} \circ \psi$, where $\mathcal{G} : M \to T\mathbb{R}^3|_M$ is the Gauss map defined by $\mathcal{G}(p) = (p, \vec{U}(p))$. However, for surfaces with boundary, $\operatorname{spt} N(M) \cap T_p\mathbb{R}^3$ contains more than one point for all $p \in M$. Thus there is no function G on M such that $(G \circ \psi)_\# D = E$. To avoid this problem, we will instead use a function G from $T\mathbb{R}^3|_M$ to $T\mathbb{R}^3|_M$, which we will define in Section 7.6. We then define $f : T\mathcal{U} \to T\mathbb{R}^3$ by the following commutative diagram:



where $\mathcal{D} = \{(\psi(p), n) : (p,n) \in \operatorname{spt} N(T)\}$. In other words, $f(p,n) = G(\psi(p), n)$ for all $(p,n) \in T\mathcal{U}$. By the Homotopy Formula for currents [2, 4.1.9],

$$D - E = D - f_\# D = h_\#([0,1] \times D) + \partial h_\#([0,1] \times D),$$

so

$$\mathbf{F}[D - f_\# D] \le \mathbf{M}[h_\#([0,1] \times D)] + \mathbf{M}[\partial h_\#([0,1] \times D)].$$

Note that f and h will be Lipschitz maps but are not differentiable on $T\mathcal{U}$. Thus far we have only considered the pushforward of currents for smooth maps between manifolds, so in Section 7.5 we will discuss the pushforward of currents for Lipschitz maps. In Section 7.5, we will use [2, 4.1.9] to show that

$$\mathbf{F}[D - f_\# D] \le \|f - \mathrm{Id}\|_{\operatorname{spt} N(T)} \max\{1, \Lambda^2\}(\mathbf{M}[D] + \mathbf{M}[\partial D]),$$

where Id denotes the identity map and Λ is the Lipschitz constant for f. In Section 7.6 we will compute $\|f - \mathrm{Id}\|_{\operatorname{spt} N(T)}$ and Λ.

The norms of the curvature 2-forms and their derivatives will be computed in Section 7.7. We will then combine $\mathbf{F}[D-E]$ and the norms of the curvature 2-forms to obtain Theorem 13.

In Section 7.8, we will examine our explicit choice of ψ. We will verify that our choice of ψ is well-defined. We will then prove Theorem 14 by computing δ_B and λ for our specific choice of ψ and then applying Theorem 13.

7.5 Pushforward of currents for Lipschitz maps

We shall first define $f_\# S$ for a flat current S and a Lipschitz map f. Let S be a flat current supported on an open set $\mathcal{V} \subseteq \mathbb{R}^m$. Let $f : \mathcal{V} \to \mathbb{R}^m$ be a Lipschitz map. We define $f_\# S$ as in [2, 4.1.14] to be the unique flat current such that if $\mathcal{W}$ is an open subset with

$\operatorname{spt} S \subset \mathcal{W} \subseteq \mathcal{V}$ and $f_i : \mathcal{W} \to \mathbb{R}^m$ is a sequence of smooth maps such that $f_i \to f$ uniformly on $\mathcal{W}$, and $\|Df_i\|$ is bounded, then

$$\lim_{i\to\infty} \mathbf{F}[f_\# T - f_{i\#} T] = 0.$$

Note that given integers a_j and closed, compact, oriented manifolds M_i of the same dimension,

$$f_\# \left(\sum_{j=1}^{t} a_j [M_j] \right) = \sum_{j=1}^{t} a_j [f(M_j)]. \tag{16}$$

We can construct the smooth maps f_i from the Lipschitz map f as follows. Let $B(p, r) \subset \mathbb{R}^m$ be the open ball of radius r centered at p. Define $\widetilde{\varphi} : \mathbb{R}^3 \to \mathbb{R}$ to be a radially symmetric smooth function that is supported on $B(0, 1)$ and normalized so that $\int_{\mathbb{R}^3} \varphi(x, 1)dx = 1$. Define $\varphi_\epsilon : \mathbb{R}^3 \to \mathbb{R}$ for $\epsilon > 0$ by $\varphi_\epsilon(x) = \frac{1}{\epsilon^3}\widetilde{\varphi}(\frac{x}{\epsilon})$. Note φ_ϵ is supported on $B(0, \epsilon)$ with $\int_{\mathbb{R}^3} \varphi_\epsilon(x)dx = 1$. We define $f_\epsilon : \mathcal{V} \to \mathbb{R}^m$ for $\epsilon > 0$ using convolution by

$$f_\epsilon(x) = (\varphi_\epsilon * f)(x) = \int_{\mathbb{R}^3} \varphi_\epsilon(x - y) f(y) dV(y).$$

Clearly f_ϵ is defined on an open subset $\mathcal{W} \subset \mathcal{V}$ for ϵ sufficiently small. If f is a Lipschitz function with Lipschitz constant Λ, we have

$$\begin{aligned}
\|f_\epsilon(x) - f(x)\| &= \left\| \int_{\mathbb{R}^3} \varphi_\epsilon(x - y) f(y) dV(y) - f(x) \right\| \\
&= \left\| \int_{\mathbb{R}^3} \varphi_\epsilon(x - y)(f(y) - f(x)) dV(y) \right\| \\
&\leq \int_{\mathbb{R}^3} \varphi_\epsilon(x - y) \|f(y) - f(x)\| \, dV(y) \\
&\leq \int_{\mathbb{R}^3} \varphi_\epsilon(x - y) \Lambda \|y - x\| \, dV(y) \\
&\leq \Lambda \int_{\mathbb{R}^3} \varphi_\epsilon(x - y) \epsilon dV(y) \\
&= \Lambda\epsilon.
\end{aligned}$$

Therefore, $f_\epsilon \to f$ uniformly. Each f_ϵ is smooth since $Df_\epsilon = D\varphi_\epsilon * f$. Also,

$$\begin{aligned}
\left\| \frac{f_\epsilon(x + h\vec{v}) - f_\epsilon(x)}{h} \right\| &= \left\| \frac{1}{h} \int_{\mathbb{R}^3} \varphi_\epsilon(y) f(x + h\vec{v} - y) dV(y) - \frac{1}{h} \int_{\mathbb{R}^3} \varphi_\epsilon(y) f(x - y) dV(y) \right\| \\
&= \left\| \int_{\mathbb{R}^3} \varphi_\epsilon(y) \frac{f(x + h\vec{v} - y) - f(x - y)}{h} dV(y) \right\| \\
&\leq \int_{\mathbb{R}^3} \varphi_\epsilon(y) \left\| \frac{f(x + h\vec{v} - y) - f(x - y)}{h} \right\| dV(y) \\
&\leq \int_{\mathbb{R}^3} \varphi_\epsilon(y) \Lambda \|\vec{v}\| dV(y) \\
&= \Lambda \|\vec{v}\|,
\end{aligned}$$

so $\|Df_\epsilon\| \leq \Lambda$. We then take $f_i = f_{\epsilon_i}$ for some ϵ_i converging to zero.

To prove Theorem 13, we will construct a Lipschitz map $f : T\mathcal{U} \to T\mathbb{R}^3$ and then derive an estimate for $\mathbf{F}[D - f_\# D]$. Given f, we estimate $\mathbf{F}[D - f_\# D]$ as follows. Consider the homotopy $h : [0, 1] \times T\mathcal{U} \to T\mathcal{U}$ defined by

$$h(t, p, n) = (1 - t)(p, n) + tf(p, n).$$

Approximate h by smooth maps $h_\epsilon : [0,1] \times \mathcal{W} \to \mathbb{R}^3$, where $\mathcal{W} \subset T\mathbb{R}^3$ is open set, defined by $h_\epsilon(t,\cdot) = \varphi_\epsilon * h(t,\cdot)$. Define $f_\epsilon(x) = h_\epsilon(x,1)$. By the Homotopy Formula for currents [2, 4.1.9],

$$D - f_{\epsilon\#}D = h_{\epsilon\#}([0,1] \times D) + \partial h_{\epsilon\#}([0,1] \times D),$$

so by [2, 4.1.9],

$$\begin{aligned} \mathbf{F}[D - f_{\epsilon\#}D] &= \mathbf{M}[h_{\epsilon\#}([0,1] \times D)] + \mathbf{M}[\partial h_{\epsilon\#}([0,1] \times D)] \\ &\leq \|f_\epsilon - \mathrm{Id}\|_{\mathrm{spt}\,D} \sup\{1, \|Df_\epsilon\|^2\}(\mathbf{M}[D] + \mathbf{M}[\partial D]). \\ &\leq \|f_\epsilon - \mathrm{Id}\|_{\mathrm{spt}\,D} \max\{1, \Lambda^2\}(\mathbf{M}[D] + \mathbf{M}[\partial D]). \end{aligned}$$

where Λ is the Lipschitz constant for f. By the triangle inequality,

$$\begin{aligned} \mathbf{F}[D - f_{\#}D] &\leq \mathbf{F}[D - f_{\epsilon\#}D] + \mathbf{F}[f_{\epsilon\#}D - f_{\#}D] \\ &\leq \|f_\epsilon - \mathrm{Id}\|_{\mathrm{spt}\,D} \max\{1, \Lambda^2\}(\mathbf{M}[D] + \mathbf{M}[\partial D]) + \mathbf{F}[f_{\epsilon\#}D - f_{\#}D]. \end{aligned}$$

for all $\epsilon > 0$. So letting $\epsilon \to 0^+$, we obtain

$$\mathbf{F}[D - f_{\#}D] \leq \|f - \mathrm{Id}\|_{\mathrm{spt}\,D} \max\{1, \Lambda^2\}(\mathbf{M}[D] + \mathbf{M}[\partial D]).$$

7.6 Definition and computations for f

We will define $f : T\mathcal{U} \to T\mathbb{R}^3$ in terms of a generalization $G : T\mathbb{R}^3|_M \to T\mathbb{R}^3|_M$ of the Gauss map. We first need the following lemma.

Lemma 2. *Let $(\vec{T}, \vec{V}, \vec{U})$ be the Darboux frame on ∂M. We can extend this frame field onto an open neighborhood W of ∂M in M such that $\vec{U}$ is the unit normal vector field on M and $\nabla_{\vec{V}}\vec{V}$ is parallel to $\vec{U}$ on ∂M.*

Proof. By the Collaring Theorem [4, p. 113, 152], there is an open neighborhood W of ∂M in M and a smooth embedding $\mathbf{c} : \partial M \times [0,\infty) \to W$ such that $\mathbf{c}(p,0) = p$ and $\frac{\partial \mathbf{c}}{\partial t}(p,0) = \vec{V}(p)$ (where t is the parameter on $[0,\infty)$) for all $p \in \partial M$. At each point $p \in \partial M$, define the curve $\gamma_p : [0,\infty) \to M$ by $\gamma_p(t) = \mathbf{c}(p,t)$. Using parallel transport, we can extend V to a vector field on W such that $\vec{V}$ is tangent to M on W and for each $p \in \partial M$, $\nabla_{\gamma_p'}\vec{V}$ is parallel to $\vec{U}$ along γ_p [5, p. 60]. In particular, since $\gamma_p'(p,0) = \vec{V}(p)$, $\nabla_{\vec{V}}\vec{V}$ is parallel to $\vec{U}$ on ∂M. Finally, extend $\vec{T}$ to a vector field on W so that $(\vec{T}, \vec{V}, \vec{U})$ form a positively oriented orthonormal frame on W. □

Definition. Let $G : T\mathbb{R}^3|_M \to T\mathbb{R}^3|_M$ be defined as follows. Let $\epsilon > 0$. Let $(\vec{T}, \vec{V}, \vec{U})$ be an extension of the Darboux frame on ∂M defined in Lemma 2. If $p \in M \setminus W$, let

$$G(p,n) = \begin{cases} (p, +\vec{U}(p)) & \text{if } n \bullet \vec{U} \geq 0, \\ (p, -\vec{U}(p)) & \text{if } n \bullet \vec{U} < 0. \end{cases}$$

If $p \in W$, let $\mathbf{n} : [0,\infty) \times [-\pi/2, \pi/2] \times [0, 2\pi] \to T_p\mathbb{R}^3$ be a spherical coordinate parameterization of $T_p\mathbb{R}^3$ defined by

$$\mathbf{n}(\rho,\varphi,\theta) = \rho\cos\varphi\sin\theta\vec{T} + \rho\cos\varphi\cos\theta\vec{V} + \rho\sin\varphi\vec{U}.$$

Define $g : [-\pi/2, \pi/2] \to [-\pi/2, \pi/2]$ by

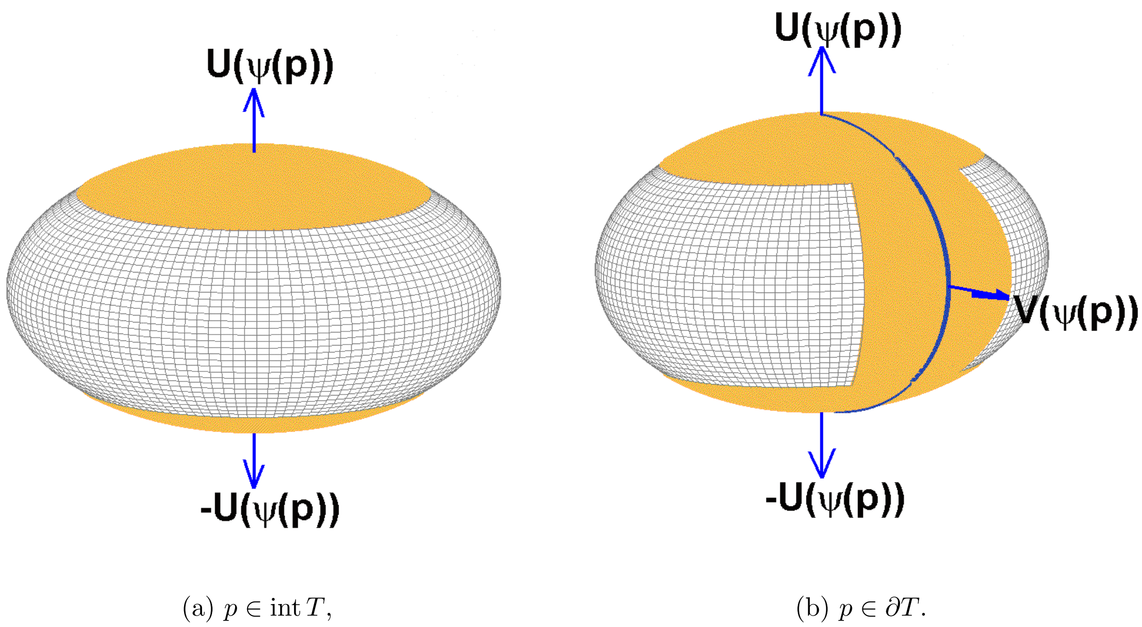
$$g(\varphi) = \begin{cases} \frac{-\pi}{2} & \text{if } \varphi < -\pi/2 + \alpha_{\vec{U},B} - \epsilon, \\ \frac{\pi}{\pi - 2\alpha_{\vec{U},B} - \epsilon}\varphi & \text{if } |\varphi| \leq \pi/2 - \alpha_{\vec{U},B} - \epsilon, \\ \frac{\pi}{2} & \text{if } \varphi > \pi/2 - \alpha_{\vec{U},B} + \epsilon, \end{cases}.$$

Let $G(p, \mathbf{n}(\rho, \varphi, \theta)) = (p, \mathbf{n}(1, g(\varphi), 0))$ if $\rho > 0$, and $G(p, 0) = (p, 0)$.

Recall $f : T\mathcal{U} \to T\mathbb{R}^3$ is defined by $f(p, n) = G(\psi(p), n)$ for all $(p, n) \in T\mathcal{U}$.

To effectively use f, we need Lemmas 3 and 4 below, which describe how $\operatorname{spt} N(T)$ and $\operatorname{spt} N(M)$ are positioned relative to each other in $T\mathbb{R}^3$. From these lemmas, we will be able to establish $f_\# D = E$ and compute $\|f - \mathrm{Id}\|_{\operatorname{spt} N(T)}$ and Λ. Recall that $\alpha_{\vec{U},B}$ is the supremum over $x \in T \cap B$ of the angles between unit normal vectors to those faces whose closure contains x and the unit normal vector to M at $\psi(x)$. Also recall that if $\partial T \cap B = \emptyset$, then $\alpha_{\vec{T},B} = 0$, and if $\partial T \cap B \neq \emptyset$, then $\alpha_{\vec{T},B}$ is the supremum over $x \in \partial T \cap B$ of the angles between edges whose closure contains x and the tangent vector to ∂M at $\psi(x)$. For $p \in \operatorname{int} T \cap B$, let A_p be the set of all unit vectors n that lie within an angular distance $\alpha_{\vec{U},B}$ from one of the unit normal vectors $\vec{U}(\psi(p))$ or $-\vec{U}(\psi(p))$. Geometrically, for $p \in \operatorname{int} T \cap B$, A_p is the union of the spherical caps centered at $\pm\vec{U}(\psi(p))$ of angular radius $\alpha_{\vec{U},B}$ (see Figure 6(a)). For $p \in \partial T \cap B$, let A_p be the union of the set of all unit vectors n that lie within an angular distance $\alpha_{\vec{U},B}$ from one of the unit normal vectors $\vec{U}(\psi(p))$ or $-\vec{U}(\psi(p))$ and the set $\{\mathbf{n}(1, \varphi, \theta) : |\theta| < \alpha_{\vec{T},B}\}$. Geometrically, for $p \in \partial T \cap B$, A_p union of the same spherical caps described above and a spherical wedge whose vertices are $\vec{U}(\psi(p))$ and $-\vec{U}(\psi(p))$ and whose boundary curves are a distance $\alpha_{\vec{T},B}$ from the geodesic arc joining $-\vec{U}(\psi(p))$ to $\vec{U}(\psi(p))$ via $\vec{V}(\psi(p))$ (see Figure 6(b)). We claim that $\operatorname{spt} N(T) \cap T_p\mathbb{R}^3 \subseteq A_p$ for all $p \in T \cap B$.

Figure 6: The set A_p for



(a) $p \in \operatorname{int} T$, (b) $p \in \partial T$.

Lemma 3. *Suppose $\alpha_{\vec{U},B} < \pi/2$. Let $p \in \operatorname{int} T \cap B$ and recall A_p be the set of all points $(p, n) \in T_p\mathbb{R}^3$ such that n lies within an angular distance $\alpha_{\vec{U},B}$ from one of the unit normal vectors $\vec{U}(\psi(p))$ or $-\vec{U}(\psi(p))$. Then* $\operatorname{spt} N(T) \cap T_p\mathbb{R}^3 \subseteq A_p$.

Proof. We shall first establish some geometric facts. By the definition of $\alpha_{\vec{U},B}$, the unit normal vectors to the faces of T containing p lie in A_p. Furthermore, geodesic arcs joining the positive normal vectors to T together lie in the spherical cap containing $\vec{U}(\psi(p))$ and the geodesic arcs joining together negative normal vectors to T lie in the spherical cap containing $-\vec{U}(\psi(p))$ (see Figure 6(a)). Now the cases where p lies on the interior of a face or interior edge are easy. If p lies on the interior of a face of T, then $\operatorname{spt} N(T) \cap T_p\mathbb{R}^3$ consists of the two normal vectors to the face containing p and both normal vectors lie in A_p. If p lies on the interior of an edge of T, then $\operatorname{spt} N(T) \cap T_p\mathbb{R}^3$ consists of two geodesic arcs, one joining two positive normal vectors to T and the other joining two negative normal vectors to T, and thus $\operatorname{spt} N(T) \cap T_p\mathbb{R}^3 \subseteq A_p$.

Finally consider the case where p is an interior vertex of T and consider the normal cycle of T above p. We shall first show $\operatorname{spt} \partial N_p(T) \subseteq A_p$. By Theorem 1,

$$\partial N_p(T) = \partial(N(T) \llcorner \pi^{-1}(p)) = \sum_{f \subset T \text{ a face}} \partial(N(f) \llcorner \pi^{-1}(p)) - \sum_{e \subset \operatorname{int} T \text{ an edge}} \partial(N(e) \llcorner \pi^{-1}(p)).$$

We know $\partial(N(e) \llcorner \pi^{-1}(p))$ for an edge e incident at p is the great circle of vectors perpendicular to e. Also $\partial(N(f) \llcorner \pi^{-1}(p))$ for a face f incident at p consists of two geodesic arcs joining the positive and negative normal vectors to f and perpendicular to the edges incident at f (see Figure 7). Thus $\partial N_p(T)$ consists of geodesic arcs perpendicular to the edges of T incident at p. If we consider the great circle Γ perpendicular to a particular edge e incident at p and the faces f_1 and f_2 incident to e, then

$$(\partial(N(f_1) \llcorner \pi^{-1}(p)) + \partial(N(f_2) \llcorner \pi^{-1}(p)) - \partial(N(e) \llcorner \pi^{-1}(p))) \llcorner \Gamma$$

consists of two geodesic arcs, one joining two positive normal vectors to T and the other joining two negative normal vectors to T (see Figure 7).

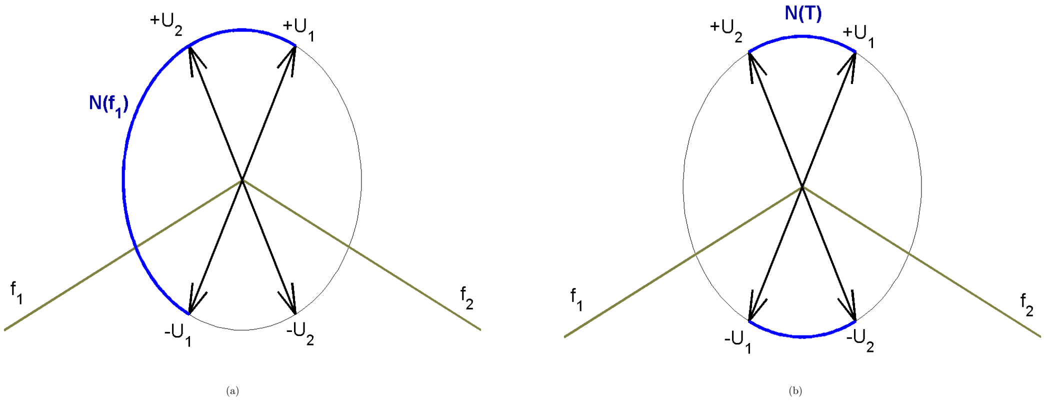
Hence $\partial N_p(T)$ consists of geodesic arcs joining either two positive normal vectors to T and the other joining two negative normal vectors to T. These geodesic arcs lie in A_p and thus $\operatorname{spt} \partial N_p(T) \subseteq A_p$.

Let $S_p\mathbb{R}^3$ denote the space of all unit vectors in $T_p\mathbb{R}^3$. By the Constancy Theorem, $\operatorname{spt}(N_p(T) - c[S_p\mathbb{R}^3 \setminus A_p]) \subseteq A_p$ for some integer c. We claim $c = 0$. Fix a vector $n \in S_p\mathbb{R}^3$ such that $n \bullet \vec{U}(\psi(p)) = 0$. Let η be a 2-form on $S_p\mathbb{R}^3$ supported on a small neighborhood of n with $\langle [S_p\mathbb{R}^3], \eta \rangle = 1$. Then $c = \langle N_p(T), \eta \rangle$. If the support of η is sufficiently small, then by Theorem 1,

$$\begin{aligned} c &= \sum_{f \subset T \text{ a face}} \langle N(f) \llcorner \pi^{-1}(p), \eta \rangle - \sum_{e \subset \operatorname{int} T \text{ an edge}} \langle N(e) \llcorner \pi^{-1}(p), \eta \rangle + \langle N(p), \eta \rangle \\ &= \#\{f \subset T : (p, n) \in \operatorname{spt} N(f)\} - \#\{e \subset T : (p, n) \in \operatorname{spt} N(e)\} + 1. \end{aligned} \tag{17}$$

We know (p, n) lies in the support of the normal cycle of a face or edge incident to p if and only if the face or edge is a subset of $H = \{q \in \mathbb{R}^3 : q \bullet n \le 0\}$ [10]. Let $\operatorname{Star}(p, T)$ denote the union of $\{p\}$ and the interiors of all faces and edges incident to p. Then (17) is equivalent to the statement that c equals the Euler characteristic of $\operatorname{Star}(p, T) \cap H$. Parameterize $\mathbb{R}^3$ using polar coordinates (r, θ, z), letting p be the origin, n be $(1, 0, 0)$, and $\vec{U}(\psi(p))$ be $(0, 0, 1)$. Since $\alpha_{\vec{U},B} < \pi/2$, it follows that $\vec{u} \bullet \vec{U}(\psi(p)) > 0$ for any positive normal vector

Figure 7: Consider the circle Γ of vector perpendicular to an edge. For each face incident to the edge, the portion of the normal cycle of the face lying in Γ is a geodesic arc between the unit normal vectors to the face (see (a)). The portion of the normal cycle of the edge lying in Γ is a current associated with all of Γ. Summing these together, we obtain the portion of $N(T)$ lying in Γ (see (b)), which consists of geodesic arcs joining to positive unit normal vectors to T and joining to negative unit normal vectors to T.



$\vec{u}$ to a face of T incident at p. Thus the θ-coordinate of a unit speed parameterization of $\partial\,\mathrm{Star}(p,T)$ is a monotone function (see Figure 8). So the projection of $\mathrm{Star}(p,T)$ onto the place given by $z = 0$ is a one-to-one map whose range is an open neighborhood of p that is topologically equivalent to a disk. Hence $\mathrm{Star}(p,T) \cap H$ is topologically equivalent to $\{(r,\theta,0) : 0 \le r < 1\} \cap H$, which has Euler characteristic zero. Therefore $c = 0$ and $\mathrm{spt}\, N_p(T) \subseteq A_p$.

□

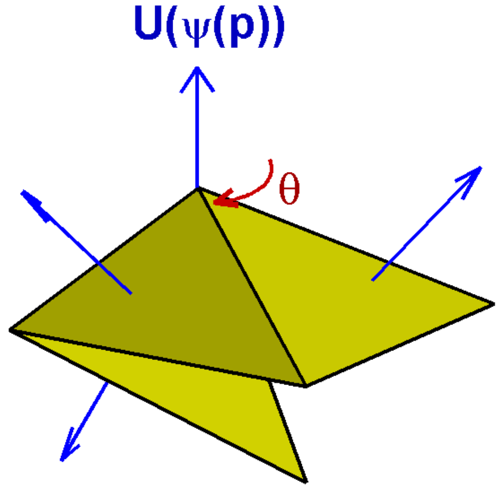
Lemma 4. *Suppose $\alpha_{\vec{U},B} < \pi/2$ and $\alpha_{\vec{T},B} < \pi/2$. Further suppose that $\vec{v} \bullet D\psi(\vec{v}) \ge 0$ for all vectors $\vec{v} \in T_x\mathbb{R}^3$ and for almost all points x in a neighborhood of ∂T in T. Let $p \in \partial T \cap B$ and recall that A_p is the union of the set of all unit vectors n that lie within an angular distance $\alpha_{\vec{U},B}$ from one of the unit normal vectors $\vec{U}(\psi(p))$ or $-\vec{U}(\psi(p))$ and the set $\{\mathbf{n}(1,\varphi,\theta) : |\theta| < \alpha_{\vec{T},B}\}$ (see Figure 6(b)). Then $\mathrm{spt}\, N(T) \cap T_p\mathbb{R}^3 \subseteq A_p$.*

Proof. We shall use a similar proof as before. If p lies on the interior of an edge e of ∂T, $\mathrm{spt}\, N(T) \cap T_p\mathbb{R}^3$ is a geodesic arc between the positive and negative normal vectors to the face f incident to e passing through the unit vector $-\vec{V}$ perpendicular to e and tangent to f. By the defintions of $\alpha_{\vec{U},B}$ and $\alpha_{\vec{T},B}$, the normal vector to f and $-\vec{V}$ lie in A_p and thus the geodesic arc lies in A_p, i.e. $\mathrm{spt}\, N(T) \cap T_p\mathbb{R}^3 \subseteq A_p$.

Suppose p is a vertex of T and consider the normal cycle of T above p. We shall first show $\mathrm{spt}\, N_p(T) \subseteq A_p$. As before, $\partial N_p(T)$ consists of geodesic arcs perpendicular to the edges of T incident at p. All but two of these geodesic arcs join the positive normal vectors to T with each other and the negative normal vectors to T together. The remaining two geodesic arcs are perpendicular to the boundary edges of T incident at p, joining the unit normal vectors to the face f incident to e and passing through the unit vector $-\vec{V}$ perpendicular to e and tangent to f. All these geodesic arcs lie in A_p, so $\mathrm{spt}\, \partial N_p(T) \subseteq A_p$.

By the Constancy Theorem, $\mathrm{spt}(N_p(T) - c[S_p\mathbb{R}^3 \setminus A_p]) \subseteq A_p$ for some integer c, which

Figure 8: Let $\vec{u}$ be a unit normal vector to a face of T incident to p. If $\vec{u} \bullet \vec{U}(\psi(p)) > 0$, then θ is increasing. Otherwise, θ is constant or decreasing.



we claim to be zero. Consider $n = \vec{T}(\psi(p))$. We compute

$$c = \#\{f \subset T \text{ a face} : (p,n) \in \operatorname{spt} N(f)\} - \#\{e \subset \operatorname{int} T \text{ an edge} : (p,n) \in \operatorname{spt} N(e)\}. \quad (18)$$

The point (p, n) lies in the support of the normal cycle of a face or edge incident to p if and only if the face or edge is a subset of $H = \{q \in \mathbb{R}^3 : q \bullet n \le 0\}$. Let $\operatorname{Star}(p, T)$ denote the union of the interior of all faces and interior edges incident to p. Then (18) is equivalent to the statement that c equals the Euler characteristic of $\operatorname{Star}(p, T) \cap H$.

Parameterize $\mathbb{R}^3$ using polar coordinates (r, θ, z), letting p be the origin, n be $(1, 0, 0)$, and $\vec{U}(\psi(p))$ be $(0, 0, 1)$. As before, the θ coordinate of a unit speed parameterization of $\partial \operatorname{Star}(p, T)$ is a monotone function. Suppose that the θ coordinate of $\partial \operatorname{Star}(p, T)$ increases from θ_1 to θ_2. We want to show that $\theta_2 - \theta_1 < 2\pi$. However, this is not immediately obvious since $\partial \operatorname{Star}(p, T)$ is not a closed curve as in the case that $p \in \operatorname{int} M$. In fact, we may have a situation as Figure 9 where the θ-coordinate for a parameterization of $\partial \operatorname{Star}(p, T)$ increases by 3π between the two endpoints of $\partial \operatorname{Star}(p, T)$. If we follow this proof for the situation in Figure 9, we find that $c = 1$. So to prove that $c = 0$, we must use the condition that $\vec{v} \bullet D\psi(\vec{v}) \ge 0$ for all vectors $\vec{v} \in T_x\mathbb{R}^3$ and for almost all points x in a tubular neighborhood of ∂T. By the definition of $\alpha_{\vec{T},B}$, we may let $-\alpha_{\vec{T},B} < \theta_1 < \alpha_{\vec{T},B}$ and $(2m+1)\pi - \alpha_{\vec{T},B} < \theta_2 < (2m+1)\pi + \alpha_{\vec{T},B}$ for some integer $m \ge 0$. For every $\varepsilon > 0$, there is a $\delta > 0$ such that if $\|y - \psi(p)\| < \delta$ for $y \in M$, then the θ coordinate of y lies on $(-\varepsilon, \pi + \varepsilon)$. Consider $x \in \operatorname{Star}(p, T)$ such that $\|\psi(x) - \psi(p)\| < \delta$. Then

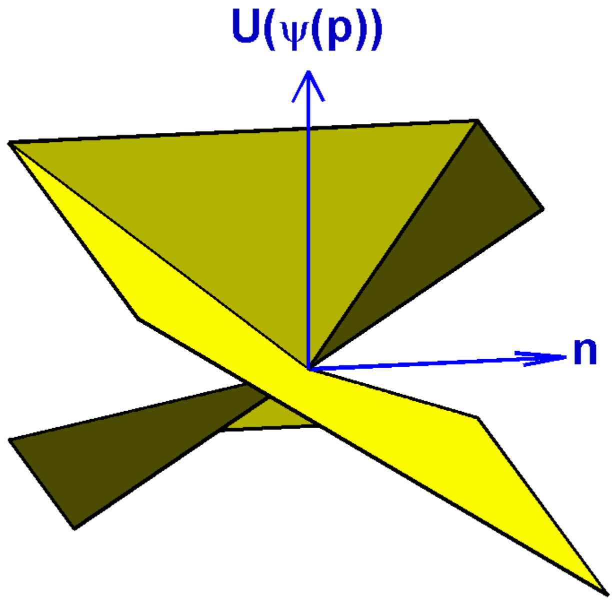
$$(x - p) \bullet (\psi(x) - \psi(p)) = \int_\ell (x - p) \bullet D\psi\left(\tfrac{x-p}{\|x-p\|}\right) ds \ge 0,$$

where ℓ is the line segment from p to x. Thus the angle between $x - p$ and $\psi(x) - \psi(p)$ is at most $\pi/2$. Hence we may regard the θ coordinate of $\operatorname{Star}(p, T)$ as increasing from at least $-\pi/2 - \varepsilon$ to at most $3\pi/2 + \varepsilon$. Thus $\theta_2 - \theta_1 < 2\pi + 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\theta_2 - \theta_1 < 2\pi$. Since

$$2m\pi < (2m+1)\pi - 2\alpha_{\vec{T},B} \le \theta_2 - \theta_1 \le 2\pi,$$

$m = 0$. Hence $\theta_2 - \theta_1 < (2m+1)\pi - 2\alpha = \pi + 2\alpha < 2\pi$. Under the projection onto the plane given by $z = 0$, $\mathrm{Star}(p, T) \cap H$ is topologically equivalent to $\{(r, \theta, 0) : 0 < r < 1, \theta_1 < \theta < \theta_2\} \cap H$, which has Euler characteristic zero. Therefore $c = 0$ and $\mathrm{spt}\, N_p(T) \subseteq A_p$.

Figure 9: Lemma 4 fails for the following triangulated surface, which is homeomorphic to a half of a circular disk with rounded corners.



□

Since G is not continuous on all of $T\mathbb{R}^3|_M$, it is not obvious that f is continuous on $\mathrm{spt}\, D$. However, G is continuous on an open neighborhood of $\{(\psi(p), n) : p \in T \cap B, n \in A_p\}$ in $T\mathbb{R}^3|_M$. Since $\mathrm{spt}\, D \subset \{(p, n) : p \in T \cap B, n \in A_p\}$ by Lemmas 3 and 4, f is continuous on an open neighborhood of $\mathrm{spt}\, D$. Since ψ is Lipschitz and G has bounded derivatives almost everywhere (see Lemma 7), f is also Lipschitz on an open neighborhood of $\mathrm{spt}\, D$. Thus we may apply the homotopy formula for currents. Recall $D = N(T) \llcorner \pi^{-1}(B)$, $E = N(M) \llcorner \pi^{-1}(\psi(B))$, and

$$\mathbf{F}[D - f_\# D] \leq \|f - \mathrm{Id}\|_{\mathrm{spt}\, N(T)} \max\{1, \Lambda^2\}(\mathbf{M}[D] + \mathbf{M}[\partial D]) \tag{19}$$

where Λ is the Lipschitz constant for f. Thus to obtain our bound on $\mathbf{F}[D - E]$, we must show $f_\# D = E$ and compute $\|f - \mathrm{Id}\|_{\mathrm{spt}\, N(T)}$ and Λ.

Lemma 5. *$f_\# D = E$.*

Proof. First suppose $\mathrm{spt}\, E \cap \pi^{-1}(\mathrm{int}\, M) \neq \emptyset$. Recall that for all $(p, n) \in T\mathbb{R}^3$, $f(p, n) = (\psi(p), u)$ for some unit vector u. Also recall that f maps positive (and negative) normal vectors to the faces of T to the positive (and negative) normal vectors to M. It follows from the definition of f and Lemmas 3 and 4 that $f(\mathrm{spt}\, D \cap T_p\mathbb{R}^3) = \mathrm{spt}\, E \cap T_{\psi(p)}\mathbb{R}^3$ for all $p \in T$; if $p \in \mathrm{int}\, T$, then $f(\mathrm{spt}\, D \cap T_p\mathbb{R}^3)$ is the set of unit vectors perpendicular to M at $\psi(p)$, and if $p \in \partial T$, then $f(\mathrm{spt}\, D \cap T_p\mathbb{R}^3)$ is the geodesic arc from $-\vec{U}(\psi(p))$ to $-\vec{V}(\psi(p))$ to $\vec{U}(\psi(p))$. Similarly, $f(\mathrm{spt}\, \partial D \cap T_p\mathbb{R}^3) = \mathrm{spt}\, \partial E \cap T_{\psi(p)}\mathbb{R}^3$ for all $p \in T$. Hence, $\mathrm{spt}\, f_\# D \subseteq \mathrm{spt}\, E$ and $\mathrm{spt}\, \partial f_\# D \subseteq \mathrm{spt}\, \partial E$. So by the Constancy Theorem, $f_\#(D \llcorner \pi^{-1}(\mathrm{int}\, T)) = cE \llcorner \pi^{-1}(\mathrm{int}\, M)$ for some $c \in \mathbb{R}$. Define the 2-form ω^A by $\omega_{(p,n)} = (u,0) \wedge (v,0)$ if (u,v,n) is a positively oriented orthonormal frame. One can show that $i_+^* \omega^A = -i_-^* \omega^A = dA$, where dA is the area form on M. Hence,

$$c = \frac{\langle f_\# D, \omega^A \rangle}{\langle E, \omega^A \rangle} = \frac{\langle \psi_\#([T] \llcorner \pi^{-1}(B)), dA \rangle}{\langle [M] \llcorner \pi^{-1}(\psi(B)), dA \rangle}.$$

Since ψ is a homeomorphism, $\psi_\#([T] \llcorner \pi^{-1}(B)) = [M] \llcorner \pi^{-1}(\psi(B))$ and thus $c = 1$.

If instead E is supported on the boundary of M, let Z be an open subset of T such that the intersection of the closure of Z and ∂T is empty. Then

$$\begin{aligned} f_\# D &= f_\#(N(T) \llcorner \pi^{-1}(B)) = f_\#(N(T) \llcorner \pi^{-1}(Z \cup B)) - f_\#(N(T) \llcorner \pi^{-1}(Z)) \\ &= N(M) \llcorner \pi^{-1}(\psi(Z \cup B)) - N(M) \llcorner \pi^{-1}(\psi(Z)) = N(M) \llcorner \pi^{-1}(\psi(B)) = E. \end{aligned}$$

Therefore, $f_\# D = E$ for all nonempty, regular Borel sets $B \subseteq T$. □

Lemma 6. $\|f - Id\|_{\operatorname{spt} D} \leq \max\{\delta_{M,B}, \alpha_{\vec{U},B} + \alpha_{\vec{T},B} + \epsilon\}$.

Proof. Let $G = (G_p, G_n)$ for functions $G_p, G_n : T\mathcal{U} \to \mathbb{R}^3$. For $(p,n) \in \operatorname{spt} D$,

$$\|f(p,n) - (p,n)\|_{T\mathbb{R}^3} = \|G(\psi(p),n) - (p,n)\|_{T\mathbb{R}^3} = \max\{\|\psi(p) - p\|, \|G_n(\psi(p),n) - n\|\}.$$

We know $\|\psi(p) - p\| \leq \delta_{M,B}$. If $p \in \operatorname{int} T$, then $n \in A_p$ by Lemma 3 and thus $\|G_n(\psi(p),n) - n\| \leq \alpha_{\vec{U},B}$ by the definition of A_p. If $p \in \partial T$, then $n \in A_p$ by Lemma 4. Let $n = \mathbf{n}(1,\varphi,\theta)$ in spherical coordinates. If $\varphi > \frac{\pi}{2} - \alpha_{\vec{U},B} - \epsilon$, then n lies within the spherical cap of vectors within an arclength distance $\alpha_{\vec{U},B}$ of $\vec{U}(\psi(p)) = G_n(\psi(p),n)$, so $\|G_n(\psi(p),n) - n\| < \alpha_{\vec{U},B} + \epsilon$. Similarly, if $\varphi < -\frac{\pi}{2} + \alpha_{\vec{U},B} + \epsilon$, then n lies within the spherical cap of vectors within an arclength distance of $-\vec{U}(\psi(p)) = G_n(\psi(p),n)$, so $\|G_n(\psi(p),n) - n\| < \alpha_{\vec{U},B} + \epsilon$. Otherwise, n lies the spherical sector $\{\mathbf{n}(1,\varphi,\theta) : |\theta| < \alpha_{\vec{T},B}\}$, in which case

$$\begin{aligned} \|G_n(\psi(p),n) - n\| &= \left\| \mathbf{n}\left(1, \frac{\pi/2}{\pi/2 - \alpha_{\vec{U},B} - \epsilon}\varphi, 0\right) - \mathbf{n}(1,\varphi,\theta) \right\| \\ &\leq \left\| \mathbf{n}\left(1, \frac{\pi/2}{\pi/2 - \alpha_{\vec{U},B} - \epsilon}\varphi, 0\right) - \mathbf{n}(1,\varphi,0) \right\| + \|\mathbf{n}(1,\varphi,0) - \mathbf{n}(1,\varphi,\theta)\| \\ &\leq \left(\frac{\pi/2}{\pi/2 - \alpha_{\vec{U},B} - \epsilon} - 1 \right) \varphi + \theta \\ &\leq \frac{\alpha_{\vec{U},B} + \epsilon}{\pi/2 - \alpha_{\vec{U},B} - \epsilon}(\pi/2 - \alpha_{\vec{U},B} - \epsilon) + \alpha_{\vec{T},B} \\ &= \alpha_{\vec{U},B} + \epsilon + \alpha_{\vec{T},B}. \end{aligned}$$

Therefore, $\|f(p,n) - (p,n)\|_{T\mathbb{R}^3} \leq \max\{\delta_{M,B}, \alpha_{\vec{U},B} + \alpha_{\vec{T},B} + \epsilon\}$. □

Whereever f is differentiable, we have

$$Df = DG \begin{pmatrix} D\psi & 0 \\ 0 & I \end{pmatrix}.$$

Since $\|D\psi\| \leq \lambda$ and $\|I\| = 1$, $\Lambda = \|DG\| \max\{\lambda, 1\}$.

Lemma 7. *Let $(p,n) \in \mathcal{D}$ with $p \in \psi(B)$. If $p \in \operatorname{int} M$, then $\|DG\| = \max\{1, \|h\|\}$. If $p \in \partial M$ and G is differentiable at (p,n), then*

$$\|DG\| \leq \max\left\{1, \|h\|, 2\sqrt{\|\kappa_g\|^2 + 2\|h\|^2} + \frac{\pi}{\pi - 2\alpha_{\vec{U},B} - \epsilon}\right\}.$$

Proof. Recall the Gauss map $\mathcal{G} : M \to T\mathbb{R}^3|_M$ is defined by $\mathcal{G}(q) = (q, \vec{U}(q))$. By [9], $\|D\mathcal{G}\| = \max\{1, \|h\|\}$. If $p \in \operatorname{int} M$ and $n \bullet \vec{U}(\psi(p)) > 0$, then locally $G = \mathcal{G} \circ \pi$ where $\pi : T\mathbb{R}^3 \to \mathbb{R}^3$ is defined by $\pi(q, \vec{v}) = q$. Thus $\|DG\| = \max\{1, \|h\|\}$. Similarly, if $p \in \operatorname{int} M$ and $n \bullet \vec{U}(\psi(p)) < 0$, $\|DG\| = \max\{1, \|h\|\}$.

Suppose $p \in \partial M$ and $n = \mathbf{n}(1, \varphi, \theta)$ in spherical coordinates. As before, if $|\varphi| > \pi/2 - \alpha_{\vec{U},B}$, then $\|DG\| = \max\{1, \|h\|\}$. If $|\varphi| = \pi/2 - \alpha_{\vec{U},B}$, then G is not differentiable at (p,n). If $|\varphi| < \pi/2 - \alpha_{\vec{U},B}$, (p,n) has a neighborhood in $\mathcal{D}$ that is contained in $T\mathbb{R}^3|_{\partial M}$. Let $(\vec{T}, \vec{V}, \vec{U})$ be the Darboux frame along ∂M. By differentiating with respect to the frame field $((\vec{T}, 0), (\vec{V}, 0), (0, \frac{\partial \mathbf{n}}{\partial \rho}), (0, \frac{\partial \mathbf{n}}{\partial \varphi}), (0, \frac{1}{\cos\varphi}\frac{\partial \mathbf{n}}{\partial \theta}))$ on $T\mathbb{R}^3|_{\partial M}$ and expressing the vectors in terms of the frame field $((\vec{T}, 0), (\vec{V}, 0), (0, \vec{T}), (0, \vec{V}), (0, \vec{U}))$ on $T\mathbb{R}^3|_{\partial M}$, we compute

$$DG = \begin{pmatrix} I & 0 \\ (A\hat{n}|B\hat{n}) & C \end{pmatrix}$$

where

$$A = \begin{pmatrix} 0 & \kappa_g & h(\vec{T},\vec{T}) \\ -\kappa_g & 0 & h(\vec{T},\vec{V}) \\ -h(\vec{T},\vec{T}) & -h(\vec{T},\vec{V}) & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 & h(\vec{T},\vec{V}) \\ 0 & 0 & h(\vec{V},\vec{V}) \\ -h(\vec{T},\vec{V}) & -h(\vec{V},\vec{V}) & 0 \end{pmatrix},$$

$$\hat{n} = \begin{pmatrix} 0 \\ \cos(g(\theta)) \\ \sin(g(\theta)) \end{pmatrix}, \text{ and } C = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\sin(g(\theta))g'(\theta) & 0 \\ 0 & \cos(g(\theta))g'(\theta) & 0 \end{pmatrix}.$$

Recall we may write any vector on $T\mathbb{R}^3|_{\partial M}$ as $(\vec{v}, \vec{w})$ for $\vec{v} \in T(\partial M)$ and $\vec{w} \in T\mathbb{R}^3$. Thus

$$\|DG\| = \max\{1, \sup\{\|(A\hat{n}|B\hat{n}) \cdot \vec{v} + C\vec{w}\| : \|\vec{v}\| = \|\vec{w}\| = 1\} \leq \max\{1, \|A\| + \|B\| + \|C\|\}$$

The norm of A is maximum absolute value of its eigenvalues. We compute

$$\det(A - \lambda I) = -\lambda^3 - (\kappa_g^2 + h(\vec{T},\vec{T})^2 + h(\vec{T},\vec{V})^2)\lambda,$$

so the eigenvalues of A are zero and $\pm i\sqrt{\kappa_g^2 + h(\vec{T},\vec{T})^2 + h(\vec{T},\vec{V})^2}$. Hence

$$\|A\| = \sqrt{\kappa_g^2 + h(\vec{T},\vec{T})^2 + h(\vec{T},\vec{V})^2}.$$

Similarly, $\|B\| = \sqrt{h(\vec{T},\vec{V})^2 + h(\vec{V},\vec{V})^2}$. We compute that $\|C\| = \|g'\|_\infty = \frac{\pi}{\pi - 2\alpha_{\vec{U},B} - \epsilon}$. Since $\|C\| \geq 1$,

$$\|DG\| \leq \|A\| + \|B\| + \|C\| \leq 2\sqrt{\|\kappa_g\|^2 + 2\|h\|^2} + \frac{\pi}{\pi - 2\alpha_{\vec{U},B} - \epsilon}$$

at $p \in \partial M$. □

Letting $f_\# D = E$ and substituting $\|G - \mathrm{Id}\|_{\mathrm{spt}\, N(T)}$ and $\Lambda = \|DG\|_{\mathcal{D}\cap\pi^{-1}(\psi(B))} \max\{1, \lambda\}$ into (19) yields

$$\begin{aligned}\mathbf{F}[D - E] \;\leq\; & \sup\{\delta_B, \alpha_{\vec{U},B} + \alpha_{\vec{T},B} + \epsilon\} \,(\max\{1,\lambda\} \\ & \max\left\{1, \|h\|_{M\cap\psi(B)}, 2\sqrt{\|\kappa_g\|^2_{\partial M\cap\psi(B)} + 2\|h\|^2_{\partial M\cap\psi(B)}} + \frac{\pi\sigma_{\partial M\cap\psi(B)}}{\pi - 2\alpha_{\vec{U},B} - \epsilon}\right\}\Bigg)^2 \\ & \left(\mathbf{M}[N(T)\llcorner\pi^{-1}(B)] + \mathbf{M}[\partial(N(T)\llcorner\pi^{-1}(B))]\right).\end{aligned}$$

Since $\epsilon > 0$ is arbitrary,

$$\begin{aligned}\mathbf{F}[D - E] \;\leq\; & \sup\{\delta_B, \alpha_{\vec{U},B} + \alpha_{\vec{T},B}\} \,(\max\{1,\lambda\} \\ & \max\left\{1, \|h\|_{M\cap\psi(B)}, 2\sqrt{\|\kappa_g\|^2_{\partial M\cap\psi(B)} + 2\|h\|^2_{\partial M\cap\psi(B)}} + \frac{\pi\sigma_{\partial M\cap\psi(B)}}{\pi - 2\alpha_{\vec{U},B}}\right\}\Bigg)^2 \\ & \left(\mathbf{M}[N(T)\llcorner\pi^{-1}(B)] + \mathbf{M}[\partial(N(T)\llcorner\pi^{-1}(B))]\right). \qquad (20)\end{aligned}$$

7.7 Computing norms of curvature forms

Lemma 8. *Let ω^G be the Gauss curvature 2-form. Then $\|\omega^G\| = 1$ and $\|d\omega^G\| = 3$. Hence $C(\phi^G) = 3$.*

Proof. We write $(p, n) \in T\mathbb{R}^3$ as $p = (p_1, p_2, p_3)$ and $n = (n_1, n_2, n_3)$. Let $(\vec{u}_1, \vec{u}_2, n)$ be a positively oriented frame. By the Lagrange identity [12, p. 212], for vectors $\vec{v} = (\vec{v}_1, \vec{v}_2), \vec{w} = (\vec{w}_1, \vec{w}_2)$ on $T\mathbb{R}^3$,

$$\begin{aligned}\omega^G(\vec{v}, \vec{w}) &= \begin{vmatrix} \vec{u}_1 \bullet \vec{v}_2 & \vec{u}_2 \bullet \vec{v}_2 \\ \vec{u}_1 \bullet \vec{w}_2 & \vec{u}_2 \bullet \vec{w}_2 \end{vmatrix} = (\vec{u}_1 \times \vec{u}_2) \bullet (\vec{v}_2 \times \vec{w}_2) \\ &= n \bullet (\vec{v}_2 \times \vec{w}_2) = (n_1 dn_2 \wedge dn_3 + n_2 dn_3 \wedge dn_1 + n_3 dn_1 \wedge dn_2)(\vec{v}, \vec{w}).\end{aligned}$$

Since the norms of $\vec{v}_2$ and $\vec{w}_2$ are at most one, $|\omega^G(\vec{v}, \vec{w})| \leq 1$ with equality precisely when $(\vec{v}_2, \vec{w}_2, n)$ form an orthonormal frame. We compute

$$\begin{aligned}d\omega^G &= d(n_1 dn_2 \wedge dn_3 + n_2 dn_3 \wedge dn_1 + n_3 dn_1 \wedge dn_2) \\ &= dn_1 \wedge dn_2 \wedge dn_3 + dn_2 \wedge dn_3 \wedge dn_1 + dn_3 \wedge dn_1 \wedge dn_2 \\ &= 3 dn_1 \wedge dn_2 \wedge dn_3.\end{aligned}$$

Hence $\|d\omega^G\| = 3$. □

Lemma 9. *Let ω^H be the mean curvature 2-form. Then $\|\omega^H\| = 2$ and $\|d\omega^H\| = 6$.*

Proof. Let $(\vec{u}_1, \vec{u}_2, n)$ be a positively oriented frame. By the Lagrange identity [12, p. 212], for vectors $\vec{v} = (\vec{v}_1, \vec{v}_2), \vec{w} = (\vec{w}_1, \vec{w}_2)$ on $T\mathbb{R}^3$,

$$\begin{aligned}\omega^H(\vec{v}, \vec{w}) &= \begin{vmatrix} \vec{u}_1 \bullet \vec{v}_1 & \vec{u}_2 \bullet \vec{v}_2 \\ \vec{u}_1 \bullet \vec{w}_1 & \vec{u}_2 \bullet \vec{w}_2 \end{vmatrix} + \begin{vmatrix} \vec{u}_1 \bullet \vec{v}_2 & \vec{u}_2 \bullet \vec{v}_1 \\ \vec{u}_1 \bullet \vec{w}_2 & \vec{u}_2 \bullet \vec{w}_1 \end{vmatrix} \\ &= \begin{vmatrix} \vec{u}_1 \bullet \vec{v}_1 & \vec{u}_2 \bullet \vec{v}_1 \\ \vec{u}_1 \bullet \vec{w}_2 & \vec{u}_2 \bullet \vec{w}_2 \end{vmatrix} + \begin{vmatrix} \vec{u}_1 \bullet \vec{v}_2 & \vec{u}_2 \bullet \vec{v}_2 \\ \vec{u}_1 \bullet \vec{w}_1 & \vec{u}_2 \bullet \vec{w}_1 \end{vmatrix} \\ &= (\vec{u}_1 \times \vec{u}_2) \bullet (\vec{v}_1 \times \vec{w}_2) + (\vec{u}_1 \times \vec{u}_2) \bullet (\vec{v}_2 \times \vec{w}_1) = n \bullet (\vec{v}_1 \times \vec{w}_2 + \vec{v}_2 \times \vec{w}_1) \\ &= (n_1(dp_2 \wedge dn_3 + dn_2 \wedge dp_3) + n_2(dp_3 \wedge dn_1 + dn_3 \wedge dp_1) + \\ &\quad n_3(dp_1 \wedge dn_2 + dn_1 \wedge dp_2))(\vec{v}, \vec{w}).\end{aligned}$$

Since the norms of $\vec{v}_1$, $\vec{v}_2$, $\vec{w}_1$, and $\vec{w}_2$ are all at most one, $|\omega^H(\vec{v},\vec{w})| \leq 2$ with equality precisely when $(\vec{v}_1, \vec{w}_2, n)$ and $(\vec{v}_2, \vec{w}_1, n)$ both form orthonormal frames. We compute

$$\begin{aligned} & d\omega^H \\ = \; & d[n_1(dp_2 \wedge dn_3 + dn_2 \wedge dp_3) + n_2(dp_3 \wedge dn_1 + dn_3 \wedge dp_1) + n_3(dp_1 \wedge dn_2 + dn_1 \wedge dp_2)] \\ = \; & 2dp_1 \wedge dn_2 \wedge dn_3 + 2dp_2 \wedge dn_3 \wedge dn_1 + 2dp_3 \wedge dn_1 \wedge dn_2. \end{aligned}$$

Hence $\|d\omega^H\| \leq 6$. Since for any positively oriented orthonormal frame $(\vec{e}_1, \vec{e}_2, \vec{e}_3)$,

$$d\omega^H((\vec{e}_1,\vec{e}_1),(\vec{e}_2,\vec{e}_2),(\vec{e}_3,\vec{e}_3)) = 6,$$

$\|d\omega^H\| = 6$. □

Lemma 10. *Consider the mean curvature vector form* $n\omega^H_{(p,n)} = (n_1\omega^H_{(p,n)}, n_2\omega^H_{(p,n)}, n_3\omega^H_{(p,n)})$. *For* $i = 1,2,3$, $\|n_i\omega^H\| = 2$ *and* $\|d(n_i\omega^H)\| \leq 8$. *Hence* $C(\phi^{H\vec{U}}) \leq 12$.

Proof. Clearly since $|n_i| \leq 1$ with equality when $n_i = \pm 1$ and $n_j = 0$ for $j \neq i$, $\|n_i\omega^H\| = \|\omega^H\| = 2$. Recall $\|\eta_1 \wedge \eta_2\| \leq 3\|\eta_1\|\|\eta_2\|$ for any 1-form η_1 and 2-form η_2 [2]. We compute

$$\|d(n_i\omega^H)\| = \|dn_i \wedge \omega^H + n_i d\omega^H\| \leq 3\|dn_i\|\|\omega^H\| + |n_i|\|d\omega^H\| = 3 \cdot 1 \cdot 2 + 1 \cdot 6 = 12.$$

□

Recall from the statement of Theorem 13 and Section 7.4 that

$$|\phi(B) - \phi(\psi(B))| \leq \tfrac{1}{2}C(\phi)\mathbf{F}[D-E],$$

where

$$\begin{aligned} C(\phi^G) &= \max\{\|\omega^G\|, \|d\omega^G\|\}, \\ C(\phi^{H\vec{U}} &= \sqrt{3}\max\{\|n_1\omega^H\|, \|d(n_1\omega^H)\|\}. \end{aligned}$$

In this section, we computed $C(\phi^G) = 3$ and $C(\phi^{H\vec{U}}) = 12\sqrt{3}$. These computations combined with (20) complete the proof of Theorem 13.

7.8 Concrete example for ψ

We begin with a proof of Lemma 1: *Given a smooth surface* M *with boundary, there is an open set* $\mathcal{U}_M \subseteq \mathbb{R}^3$ *containing* M *on which we can define the map* $\mathrm{pr}_M : \mathcal{U}_M \to M$ *such that* $\mathrm{pr}_M(x)$ *is the unique closest point on* M *to* $x \in \mathcal{U}_M$. *Also, there is an open set* $\mathcal{U}_{\partial M} \subseteq \mathbb{R}^3$ *containing* ∂M *on which we can define the map* $\mathrm{pr}_{\partial M} : \mathcal{U}_{\partial M} \to \partial M$ *such that* $\mathrm{pr}_{\partial M}(x)$ *is the unique closest point on* ∂M *to* $x \in \mathcal{U}_{\partial M}$.

Proof. We shall proof that $\mathrm{pr}_{\partial M}$ is well-defined on an open neighborhood of ∂M. The proof that pr_M is well-defined is similar.

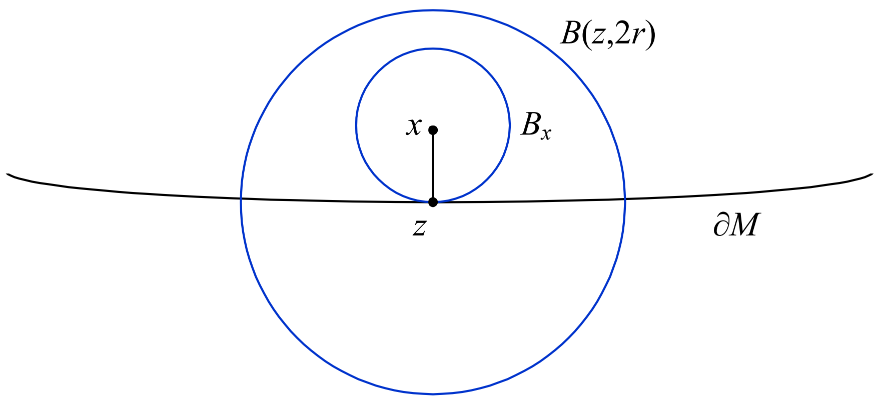
Since ∂M is a compact set, given $x \in \mathbb{R}^3$ there exists at least one point on ∂M that is closest to x. Let z be one of the closest points to x on ∂M. Let α be a local parameterization of ∂M such that $z = \alpha(0)$. Since $t = 0$ is a minimum for $\|\alpha(t) - x\|$,

$$0 = \tfrac{d}{dt}\|\alpha(t) - x\|^2|_{t=0} = 2(\alpha(0) - x) \bullet \alpha'(0),$$

so x lies on a line perpendicular to ∂M through $z = \alpha(0)$. Thus there is a closed ball B_x centered at x that is tangent to ∂M at z (see figure 10). Suppose x is within a distance $1/\|\kappa\|_{\partial M}$ of ∂M, where κ is the curvature of ∂M. Then any point on ∂M within an arc length distance $1/\|\kappa\|_{\partial M}$ of $\alpha(0)$ along α is not contained in B_x because the curvature of ∂M is bounded above by $\|\kappa\|_{\partial M}$. In other words, z is the unique closest point to x in a neighborhood of z on ∂M. However, there is the global issue that ∂M that there could be two points that are far apart along ∂M which are equally close to x due to how ∂M curves.

For $u \in \partial M$, let Γ_u be the set of points on ∂M whose arc length distance from u is greater than $1/\|\kappa\|_{\partial M}$. We want to show that if x is sufficiently close to ∂M, then all the points on Γ_z are farther away from x than z. Define $f : \partial M \to \mathbb{R}$ so that $f(u)$ is the distance from u to Γ_u. If x is closer to z that $f(z)/2$, then x is closer to z than any point on Γ_z. However, z could be any point on ∂M, so we need a lower bound on $f(z)$ that is independent of the point $z \in \partial M$. We will show such a lower bound exists by showing f is contiuous as a function on ∂M and thus attains its minimum value. Let $0 < \epsilon < 1/\|\kappa\|_{\partial M}$ and suppose $v \in \partial M$ is within a distance ϵ along ∂M of u. Then v is within a distance $f(u) + \epsilon$ of any point $w \in \Gamma_u$ that is closest to u. Since u and v are within an arclength distance ϵ along ∂M, Γ_u intersects Γ_v except on a small curve of length at most ϵ. So while w may not lie in Γ_v, there exists a point in Γ_v within a distance ϵ of w. Thus $f(v) < f(u) + 2\epsilon$. By symmetry, $f(u) < f(v) + 2\epsilon$, so $\|f(u) - f(v)\| < 2\epsilon$ whenever u and v are within an arclength distance ϵ. Therefore f is continuous and attains its minimum. Since for each $u \in \partial M$, u does not belong to the closed set Γ_u, $f > 0$. Let $f_{\min} > 0$ be the minimum value of f.

Figure 10: Given a point x near ∂M, we can find a closest point z on ∂M. We can draw a circle B_x that is centered at x and tangent to ∂M at z. If we restrict x so that $d(x, \partial M) < r$ for $r > 0$ sufficiently small, the ball $B(z, 2r)$ contains B_x and also contains only points on ∂M that are close to z in arc length distance. Of the points in $\partial M \cap B(z, 2r)$, only z is in the closure of B_x. Therefore z is the unique closest point to x on ∂M.



Now let $\mathcal{U}_{\partial M} = \{x \in \mathbb{R}^3 : d(x, M) < r\}$ where $r = \min\{1/\|\kappa\|_{\partial M}, f_{\min}/2\}$. We claim if $x \in \mathcal{U}_{\partial M}$, there is a unique closest point to x on ∂M. Let $x \in \mathcal{U}_{\partial M}$ and fix a point $z \in \partial M$ that is closest to x. Since x is within a distance r of z and the radius of B_x is less than r, $B_x \subseteq \overline{B(z, 2r)}$ (see figure 10). Since $r \leq f_{\min}/2$, $B(z, 2r) \cap \partial M$ contains only the portion of ∂M within an arc length distance $1/\|\kappa\|_{\partial M}$ from z. Thus the only point in $B_x \cap \partial M$ is z (see figure 10). Therefore z is the unique closest point to x. $\square$

Now we will compute $\delta_B = \|\psi - \mathrm{Id}\|_{T\cap B}$ and $\lambda = \sup_{x\in\mathcal{U}} \|D\psi\|$. Recall $\delta_{\partial M,B} = \sup_{x\in B} d(x, \partial M)$ and $\delta_{M,B} = \sup_{x\in B} d(x, M)$.

Lemma 11. $\delta_B \leq 2\delta_{\partial M,B} + \delta_{M,B}$

Proof. For $x \in T \cap B$,

$$\|\psi(x) - x\| = \|\operatorname{pr}_M(b(x)) - x\| \leq \|\operatorname{pr}_M(b(x)) - b(x)\| + \|b(x) - x\|.$$

If $d(x, \partial M) \geq r$, then $b(x) = x$. Suppose $d(x, \partial M) < r$. Let y be the point on the line segment from x to $\operatorname{pr}_{\partial M}(x)$ such that $d(y, \partial M) = d(\xi(x), \partial M)$. Then $x = s(x)y + (1 - s(x))\zeta(x)$ for some linear function s such that $s(y) = 1$ and $s(\zeta(x)) = 0$. But $w(y) = 1$ and $w(\zeta(x)) = 0$, so $s = w$. Thus $x = w(x)y + (1 - w(x))\zeta(x)$. Recall $b(x) = w(x)\operatorname{pr}_{\partial M}(x) + (1 - w(x))\zeta(x)$. So

$$\|b(x) - x\| = \|w(x)(\operatorname{pr}_{\partial M}(x) - y)\| \leq |w(x)| d(\xi(x), \operatorname{pr}_{\partial M}(x)) \leq \delta_{\partial M,B}.$$

Thus for $x \in M$, $d(b(x), M) \leq \|b(x) - x\| + d(x, M) \leq \delta_{\partial M,B} + \delta_{M,B}$, so $\|\operatorname{pr}_M(b(x)) - b(x)\| \leq \delta_{\partial M,B} + \delta_{M,B}$. Hence $\|\psi(x) - x\| \leq 2\delta_{\partial M,B} + \delta_{M,B}$ for all $x \in T$. In other words, $\delta_B = \|\psi - \mathrm{Id}\|_{T\cap B} \leq 2\delta_{\partial M,B} + \delta_{M,B}$. □

We know $\|D\psi\| \leq \|D\operatorname{pr}_M\|\|Db\|$, thus we need to compute $\|D\operatorname{pr}_M\|$ and $\|Db\|$ in order to compute $\|D\psi\|$. We will first compute $D\operatorname{pr}_{\partial M}$, which we will use to compute $\|D\operatorname{pr}_M\|$ and $\|Db\|$. Let $(\vec{T}, \vec{N}, \vec{B})$ be the Frenet frame on ∂M, i.e. the frame field such that $\vec{T}$ is tangent to ∂M, $\vec{T}' = \kappa\vec{N}$ for some function $\vec{N}$, and $\vec{B} = \vec{T} \times \vec{N}$. Recall κ is called the curvature of ∂M and $\vec{N}' = -\tau\vec{B}$.

Lemma 12. *Let θ be the angle between $\operatorname{pr}_{\partial M}(x) - x$ and $\vec{N}(\operatorname{pr}_{\partial M}(x))$. For all vectors $\vec{v}$,*

$$D\operatorname{pr}_{\partial M}(\vec{v}) = \frac{1}{1 - d(x, \partial M)\kappa\cos\theta}(\vec{T}(\operatorname{pr}_{\partial M}(x)) \bullet \vec{v})\vec{T}(\operatorname{pr}_{\partial M}(x)).$$

Proof. Since $\operatorname{pr}_{\partial M}$ is constant on planes perpendicular to ∂M, $D\operatorname{pr}_{\partial M}(\vec{v}) = 0$ for any vector $\vec{v}$ perpendicular to ∂M. Thus it suffices to show that

$$D\operatorname{pr}_{\partial M}(\vec{T}(\operatorname{pr}_{\partial M}(x))) = \frac{1}{1 - d(x, \partial M)\kappa\cos\theta}\vec{T}(\operatorname{pr}_{\partial M}(x)).$$

Let $\gamma : \mathbb{R} \to \partial M$ be a unit speed parameterization of ∂M and let

$$\beta = \gamma + d(x, \partial M)(\vec{N} \circ \gamma)\cos\theta + d(x, \partial M)(\vec{B} \circ \gamma)\sin\theta.$$

Note that β is a curve that passes through x. To simplify notation, we shall write $\vec{T}$ to mean $\vec{T} \circ \gamma$ and similarly for $\vec{N}$ and $\vec{B}$. We compute $\gamma' = \vec{T}$ and

$$\beta' = \vec{T} + d(x, \partial M)(-\kappa\vec{T} + \tau\vec{B})\cos\theta - d(x, \partial M)\tau\vec{N}\sin\theta.$$

Since $\operatorname{pr}_{\partial M}(\beta(s)) = \gamma(s)$ for all s, $D\operatorname{pr}_{\partial M}(\beta'(s)) = \gamma'(s)$. Thus

$$(1 - d(x, \partial M)\kappa\cos\theta)D\operatorname{pr}_{\partial M}(\vec{T}) = \vec{T} \Rightarrow D\operatorname{pr}_{\partial M}(\vec{T}) = \frac{1}{1 - d(x, \partial M)\kappa\cos\theta}\vec{T}.$$

□

Remark. We can replace the Frenet frame with a Darboux frame $(\vec{T}, \vec{V}, \vec{U})$, in which case we compute

$$D\,\mathrm{pr}_{\partial M}(\vec{T}) = \frac{1}{1 - d(x, \partial M)(\kappa_g \cos\theta + h(\vec{T}, \vec{T}) \sin\theta)} \vec{T}.$$

Lemma 13. *At a point $p \in \mathcal{U}_M$,*

$$\|D\,\mathrm{pr}_M\| \leq \max\left\{\frac{1}{1 - r\|h\|}, \frac{1}{1 - r\|\kappa\|}\right\},$$

where $r \leq \min\{1/\|h\|_M, 1/\|\kappa\|_{\partial M}\}$ is the radius of the tubular neighborhood $\mathcal{U}_M$ of T.

Proof. Consider the positively oriented, orthonormal frame field $(\vec{e}_1, \vec{e}_2, \vec{U})$ on $\mathcal{U}_M$ where at $x \in \mathcal{U}_M$, $\vec{e}_1, \vec{e}_2$ are the principal directions of M at $\mathrm{pr}_M(x)$ with principal curvatures κ_1, κ_2, respectively, and $\vec{U}$ is the unit normal vector to M at $\mathrm{pr}_M(x)$. Then at $x \in \mathcal{U}_M$ with $\mathrm{pr}_M(x) \in \mathrm{int}\, M$ [9],

$$D\,\mathrm{pr}_M = \begin{pmatrix} \frac{1}{1+d(x,M)\kappa_1} & 0 & 0 \\ 0 & \frac{1}{1+d(x,M)\kappa_2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and at $x \in \mathcal{U}_M$ with $\mathrm{pr}_M(x) \in \partial M$, $D\,\mathrm{pr}_M = D\,\mathrm{pr}_{\partial M}$, which is given by Lemma 12. Note that pr_M can only be defined if $d(x, M) \leq r < 1/|\kappa_i|$ for $i = 1, 2$ [9], so $\frac{1}{1+d(x,M)\kappa_i}$ is defined and positive for $i = 1, 2$. Hence

$$\|D\,\mathrm{pr}_M\| \leq \max\left\{\frac{1}{1 - r\|h\|}, \frac{1}{1 - r\|\kappa\|}\right\}.$$

□

We will now compute Db. Recall that for $x \in \mathcal{U}_{\partial M}$, $\xi(x)$ is the unique point such that $\mathrm{pr}_{\partial M}(\xi(x)) = \mathrm{pr}_{\partial M}(x)$, $\zeta(x) = \mathrm{pr}_{\partial M}(x) + \frac{r}{d(x,\partial M)}(x - \mathrm{pr}_{\partial M}(x))$ (see Figure 5), $w(x) = \frac{r-d(x,\partial M)}{r-d(\xi(x),\partial M)}$, and

$$b(x) = \begin{cases} \mathrm{pr}_{\partial M}(x) & \text{if } d(x, \partial M) < d(\xi(x), \partial M), \\ w(x)\,\mathrm{pr}_{\partial M}(x) + (1 - w(x))\zeta(x) & \text{if } d(\xi(x), \partial M) < d(x, \partial M) < r, \\ x & \text{if } d(x, \partial M) > r. \end{cases}.$$

At a point $x \in \mathcal{U}_M$ with $d(\xi(x), \partial M) < d(x, \partial M) < r$, let (E_1, E_2, E_3) be the positively oriented, orthonormal frame field on $\mathcal{U}_{\partial M}$ where at $x \in \mathcal{U}_{\partial M}$, $\vec{E}_1$ is tangent to the circle containing x centered at $\mathrm{pr}_{\partial M}(x)$ and perpendicular to ∂M, $\vec{E}_2$ equals the tangent vector to ∂M at $\mathrm{pr}_{\partial M}(x)$, and $\vec{E}_3$ is parallel to the vector from $\mathrm{pr}_{\partial M}(x)$ to x. This frame field has two important properties: $\vec{E}_3$ is the gradient of the map $x \mapsto d(x, \partial M)$ and $\vec{E}_1[w] = 0$ since w is constant on circles in planes perpendicular to ∂M.

Lemma 14. *Suppose $x \in \mathcal{U}_M$ with $d(\xi(x), \partial M) < d(x, \partial M) < r$. Then the matrix representation of Db in terms of the frame field $(\vec{E}_1, \vec{E}_2, \vec{E}_3)$ is*

$$\begin{aligned} Db &= \begin{pmatrix} Db(\vec{E}_1) & Db(\vec{E}_2) & Db(\vec{E}_3) \end{pmatrix} \\ &= \begin{pmatrix} W & 0 & 0 \\ 0 & (1 - W)\frac{1}{1-d(x,\partial M)\kappa\cos\theta} + W & 0 \\ 0 & -\vec{E}_2[w]r & -\vec{E}_3[w]r \end{pmatrix}, \end{aligned}$$

where $W(x) = (1 - w(x))\frac{r}{d(x,\partial M)}$.

Proof. To prove Lemma , obvserve that for all $\vec{v} \in T_x\mathbb{R}^3$,

$$Db(\vec{v}) = \vec{v}[w](\mathrm{pr}_{\partial M}(x) - \zeta(x)) + w(x)D\,\mathrm{pr}_{\partial M}(\vec{v}) + (1 - w(x))D\zeta(\vec{v}).$$

Recall that $\vec{E}_3$ is the gradient of the map $x \mapsto d(x, \partial M)$. We compute

$$\begin{aligned}
D\zeta(\vec{v}) &= D\,\mathrm{pr}_{\partial M}(\vec{v}) - \frac{r}{(d(x,\partial M))^2}\vec{v}[d(x,\partial M)](x - \mathrm{pr}_{\partial M}(x)) + \frac{r}{d(x,\partial M)}(\vec{v} - D\,\mathrm{pr}_{\partial M}(\vec{v})) \\
&= D\,\mathrm{pr}_{\partial M}(\vec{v}) - \frac{r}{(d(x,\partial M))^2}(\vec{E}_3 \bullet \vec{v})d(x,\partial M)\vec{E}_3 + \frac{r}{d(x,\partial M)}(\vec{v} - D\,\mathrm{pr}_{\partial M}(\vec{v})) \\
&= \left(1 - \frac{r}{d(x,\partial M)}\right) D\,\mathrm{pr}_{\partial M}(\vec{v}) - \frac{r}{d(x,\partial M)}(\vec{E}_3 \bullet \vec{v})\vec{E}_3 + \frac{r}{d(x,\partial M)}\vec{v}.
\end{aligned}$$

Thus

$$(1 - w(x))D\zeta(\vec{v}) = (1 - w(x) - W(x))D\,\mathrm{pr}_{\partial M}(\vec{v}) - W(x)(\vec{E}_3 \bullet \vec{v})\vec{E}_3 + W(x)\vec{v}.$$

Note that $\mathrm{pr}_{\partial M}(x) - \zeta(x) = -r\vec{E}_3$. Hence

$$\begin{aligned}
Db(\vec{v}) &= \vec{v}[w]r\vec{E}_3 + w(x)D\,\mathrm{pr}_{\partial M}(\vec{v}) + (1 - w(x) - W(x))D\,\mathrm{pr}_{\partial M}(\vec{v}) \\
&\quad -W(x)(\vec{E}_3 \bullet \vec{v})\vec{E}_3 + W(x)\vec{v} \\
&= -\vec{v}[w]r\vec{E}_3 + (1 - W(x))D\,\mathrm{pr}_{\partial M}(\vec{v}) - W(x)(\vec{E}_3 \bullet \vec{v})\vec{E}_3 + W(x)\vec{v}.
\end{aligned}$$

Recall that $D\,\mathrm{pr}_{\partial M}(\vec{E}_1) = D\,\mathrm{pr}_{\partial M}(\vec{E}_3) = 0$ and $D\,\mathrm{pr}_{\partial M}(\vec{E}_2) = \frac{1}{1-d(x,\partial M)\kappa\cos\theta}\vec{E}_2$ by Lemma 12 and that $\vec{E}_1[w] = 0$. Therefore

$$\begin{aligned}
Db(\vec{E}_1) &= W(x)\vec{E}_1, \\
Db(\vec{E}_2) &= -\vec{E}_2[w]r\vec{E}_3 + (1 - W(x))\frac{1}{1 - d(x,\partial M)\kappa\cos\theta}\vec{E}_2 + W(x)\vec{E}_2, \\
Db(\vec{E}_3) &= -\vec{E}_3[w]r\vec{E}_3 - W(x)\vec{E}_3 + W(x)\vec{E}_3 = -\vec{E}_3[w]r\vec{E}_3.
\end{aligned}$$

□

Corollary 3. *At any point $x \in \mathcal{U}_{\partial M}$ where b is differentiable,*

$$\|Db\| \leq \frac{r}{r - \delta_{\partial M,\partial T}} \sec\alpha_{\vec{T},\partial T} + \max\left\{\frac{1}{1 - r\|\kappa\|_{\partial M}}, \frac{r}{r - \delta_{\partial M,\partial T}}\right\}.$$

Before we can prove Corollary 3, we must obtain explicit bounds on W, $\vec{E}_2[w]$, and $\vec{E}_3[w]$.

Lemma 15. *If $d(\xi(x),\partial M) < d(x,\partial M) < r$, then $0 < W(x) < 1$.*

Proof. Recall that $W(x) = (1 - w(x))\frac{r}{d(x,\partial M)}$ where $w(x) = \frac{r - d(x,\partial M)}{r - d(\xi(x),\partial M)}$. We compute

$$\begin{aligned}
W(x) &= (1 - w(x))\frac{r}{d(x,\partial M)} = \frac{d(x,\partial M) - d(\xi(x),\partial M)}{r - d(\xi(x),\partial M)}\frac{r}{d(x,\partial M)} \\
&= \frac{r}{r - d(\xi(x),\partial M)}\left(1 - \frac{d(\xi(x),\partial M)}{d(x,\partial M)}\right),
\end{aligned}$$

so $W(x)$ increases as $d(x,\partial M)$ increases. Since $W(x) = 0$ when $d(x,\partial M) = d(\xi(x),\partial M)$ and $W(x) = 1$ when $d(x,\partial M) = r$, $0 < W(x) < 1$ whenever $d(\xi(x),\partial M) < d(x,\partial M) < r$. □

Lemma 16. *At any point $x \in \mathcal{U}_{\partial M}$ such that $\xi(x)$ is not a vertex of ∂T,*

$$\begin{aligned} |\vec{E}_2[w]| &\leq \frac{\sec \alpha_{\vec{T},\partial T}}{r - d(\xi(x), \partial M)}, \\ \vec{E}_3[w] &= \frac{1}{r - d(\xi(x), \partial M)}. \end{aligned}$$

Proof. Recall

$$w(x) = \frac{r - d(x, \partial M)}{r - d(\xi(x), \partial M)}.$$

Since $\vec{E}_3$ is the gradient of $x \mapsto d(x, \partial M)$, $\vec{E}_2[d(x, \partial M)] = 0$ and $\vec{E}_2[d(\xi(x), \partial M)] = \vec{E}_3 \bullet D\xi(\vec{E}_2)$ by the Chain Rule. Thus,

$$\begin{aligned} \vec{E}_2[w] &= \frac{-1}{r - d(\xi(x), \partial M)} \vec{E}_2[d(x, \partial M)] + \frac{r - d(x, \partial M)}{(r - d(\xi(x), \partial M))^2} \vec{E}_2[d(\xi(x), \partial M)] \\ &= \frac{r - d(x, \partial M)}{(r - d(\xi(x), \partial M))^2} \vec{E}_3 \bullet D\xi(E_2). \end{aligned}$$

Now we must compute $D\xi(E_2)$. Let e be the edge of ∂T containing $\xi(x)$ and let $\vec{e}$ be the unit tangent vector to e. Since $\xi(x)$ is not a vertex, $\xi(y) \in e$ for all points y near x. Therefore $D\xi(\vec{E}_2)$ is a multiple of $\vec{e}$. We know $\mathrm{pr}_{\partial M} \circ \xi = \mathrm{pr}_{\partial M}$ on $\mathcal{U}_{\partial M}$, so by Lemma 12,

$$\begin{aligned} \frac{1}{1 - d(x, \partial M)\kappa \cos\theta} \vec{E}_2 &= D\,\mathrm{pr}_{\partial M}(\vec{E}_2) \\ &= D\,\mathrm{pr}_{\partial M}(D\xi(\vec{E}_2)) = \frac{1}{1 - d(x, \partial M)\kappa \cos\theta} (\vec{E}_2 \bullet D\xi(\vec{E}_2)) \vec{E}_2. \end{aligned}$$

Hence $\vec{E}_2 \bullet D\xi(E_2) = 1$. Thus

$$D\xi(E_2) = \frac{1}{\vec{E}_2 \bullet \vec{e}} \vec{e} \;\Rightarrow\; \vec{E}_3 \bullet D\xi(E_2) = \frac{\vec{E}_3 \bullet \vec{e}}{\vec{E}_2 \bullet \vec{e}}.$$

Since the angle between $\vec{E}_2$, which is tangent to ∂M, and $\vec{e}$, which is tangent to ∂T, is at most $\alpha_{\vec{T},B} < \pi/2$, $\vec{E}_2 \bullet \vec{e} \geq \cos \alpha_{\vec{T},B}$. Since $\vec{E}_3$ and $\vec{e}$ are unit vectors, $|\vec{E}_3 \bullet \vec{e}| \leq 1$. Thus $\vec{E}_3 \bullet D\xi(E_2) \leq \sec \alpha_{\vec{T},\partial T}$. Therefore,

$$\begin{aligned} |\vec{E}_2[w]| &= \left| \frac{r - d(x, \partial M)}{(r - d(\xi(x), \partial M))^2} \frac{\vec{E}_3 \bullet \vec{e}}{\vec{E}_2 \bullet \vec{e}} \right| \leq \frac{r - d(x, \partial M)}{(r - d(\xi(x), \partial M))^2} \sec \alpha_{\vec{T},\partial T} \\ &= w(x) \frac{1}{r - d(\xi(x), \partial M)} \sec \alpha_{\vec{T},\partial T} \leq \frac{\sec \alpha_{\vec{T},\partial T}}{r - d(\xi(x), \partial M)}. \end{aligned}$$

Similarly, we compute

$$\vec{E}_3[w] = \frac{-1}{r - d(\xi(x), \partial M)} \vec{E}_3 \bullet \vec{E}_3 + \frac{r - d(x, \partial M)}{(r - d(\xi(x), \partial M))^2} \vec{E}_3 \bullet D\xi(\vec{E}_3) = \frac{-1}{r - d(\xi(x), \partial M)},$$

since $\xi(x)$ is constant along lines in $\mathcal{U}_{\partial M}$ that are perpendicular to ∂M. $\square$

We will now prove Lemma 3.

Proof. At a point $x \in \mathcal{U}_M$ with $d(x, \partial M) < d(\xi(x), \partial M)$,

$$\|Db\| = \|D\,\mathrm{pr}_{\partial M}\| \leq \frac{1}{1 - \delta_{\partial M, \partial T}\|\kappa\|_{\partial M}}$$

by Lemma 12. At $x \in \mathcal{U}_M$ with $d(x, \partial M) > r$, $\|Db\| = \|I\| = 1$. Each of these bounds are consistent with Lemma 3.

Consider $x \in \mathcal{U}_M$ with $d(\xi(x), \partial M) < d(x, \partial M) < r$. Recall

$$Db = \begin{pmatrix} W & 0 & 0 \\ 0 & (1-W)\frac{1}{1-d(x,\partial M)\kappa\cos\theta} + W & 0 \\ 0 & -\vec{E}_2[w]r & -\vec{E}_3[w]r \end{pmatrix}.$$

We can write

$$Db = \begin{pmatrix} W & 0 \\ 0 & A+B \end{pmatrix} \text{ where } A = \begin{pmatrix} 0 & 0 \\ -\vec{E}_2[w]r & 0 \end{pmatrix} \text{ and}$$

$$B = \begin{pmatrix} (1-W)\frac{1}{1-d(x,\partial M)\kappa\cos\theta} + W & 0 \\ 0 & -\vec{E}_3[w]r \end{pmatrix}.$$

By the properties of matrix norms [6, p. 283],

$$\|Db\| \leq \max\{\|W\|, \|A+B\|\} \leq \max\{1, \|A\| + \|B\|\}.$$

We compute

$$\begin{aligned}
\|A\| &= \left\| \begin{pmatrix} 0 & 0 \\ -\vec{E}_2[w]r & 0 \end{pmatrix} \right\| \leq \frac{\sec\alpha_{\vec{T},\partial T}}{r - d(\xi(x), \partial M)} r \\
&\leq \frac{r}{r - \delta_{\partial M,\partial T}} \sec\alpha_{\vec{T},\partial T}, \\
\|B\| &= \left\| \begin{pmatrix} (1-W)\frac{1}{1-d(x,\partial M)\kappa\cos\theta} + W & 0 \\ 0 & -\vec{E}_3[w]r \end{pmatrix} \right\| \\
&\leq \max\left\{ \frac{1}{1 - d(x,\partial M)\|\kappa\|_{\partial M}}, \frac{1}{r - d(\xi(x), \partial M)} r \right\} \\
&\leq \max\left\{ \frac{1}{1 - r\|\kappa\|_{\partial M}}, \frac{r}{r - \delta_{\partial M,\partial T}} \right\}.
\end{aligned}$$

Hence

$$\|A\| + \|B\| \leq \frac{r}{r - \delta_{\partial M,\partial T}} \sec\alpha_{\vec{T},\partial T} + \max\left\{ \frac{1}{1 - r\|\kappa\|_{\partial M}}, \frac{r}{r - \delta_{\partial M,\partial T}} \right\}.$$

Since the above bound on $\|A\| + \|B\|$ is greater than 1, $\|Db\|$ is bounded above by this bound on $\|A\| + \|B\|$. □

In conclusion, Theorem 13 applies to $\psi = \mathrm{pr}_M \circ b$ with

$$\begin{aligned}
\delta_B &\leq 2\delta_{\partial M,B} + \delta_{M,B}, \\
\lambda &\leq \|D\,\mathrm{pr}_M\| \|Db\| \\
&\leq \max\left\{ \frac{1}{1 - r\|h\|_M}, \frac{1}{1 - r\|\kappa\|_{\partial M}} \right\} \\
&\quad \cdot \left(\frac{r}{r - \delta_{\partial M,\partial T}} \sec\alpha_{\vec{T},\partial T} + \max\left\{ \frac{1}{1 - r\|\kappa\|_{\partial M}}, \frac{r}{r - \delta_{\partial M,\partial T}} \right\} \right).
\end{aligned}$$

References

[1] Do Carmo, Manfredo P. *Differential Geometry of Curves and Surfaces*, Englewood Cliffs, NJ: Prentice-Hall, Inc, 1976.

[2] Federer, Herbert, *Geometric Measure Theory*, Springer-Verlag, New York, NY, 1969.

[3] Fu, Joseph, "Curvature Measures of Subanalytic Sets." *American Journal of Mathematics*, Vol. 116, No. 4, August 1994, p. 819-880.

[4] Hirsch, Morris W, *Differential Topology*, Springer-Verlag, New York, NY, 1976.

[5] Lee, John M. *Riemannian Manifolds, An Introduction to Curvature*, Springer-Verlag, New York, NY, 1997.

[6] Meyer, Carl D. *Martix Analysis and Applied Linear Algebra*, Society for Industrial and Applied Mathematics, Philadelphia, PA, 2000.

[7] Meyer, Mark; Desbrun, Mathieu; Schroder, Peter; Barr, Alan H. "Discrete Differential-Geometry Operators for Triangulated 2-Manifolds, *Proceedings of the International Workshop on Visualization and Mathematics 2002*, May 2002, Berlin-Dahlem, Germany.

[8] Morgan, Frank, *Geometric Measure Theory: A Beginner's Guide*, Academic Press, San Diego, CA, 1988.

[9] Morvan, Jean-Marie and Cohen-Steiner, David, "Approximation of Normal Cycles," *Rapport de recherche*, February, 2003, INRIA, Sophia Antipolis, France.
Also see, Cohen-Steiner, David and Morvan, Jean-Marie, "Second Fundamental Measure of Geometric Sets and Local Approximation of Curvatures," Journal of Differential Geometry, (to appear).

[10] Morvan, Jean-Marie and Cohen-Steiner, David, "Restricted Delaunay Triangulations and Normal Cycle," *Symposium on Computational Geometry 2003*, June 8-10, 2003, San Diego, CA: ACM, 2003, p. 312-321.

[11] Munkres, James R, *Analysis on Manifolds*, Addison-Wesley Publishing Company, Redwood City, CA, 1991.

[12] O'Neill, Barrett, *Elementary Differential Geometry*, Academic Press, San Diego, CA, 1997.

[13] Sullivan, John M. "Curvature Measures for Discrete Surfaces," Preprint, 2003.

[14] Taubin, Gabriel, "Estimating the Tensor of Curvature of a Surface from a Polyhedral Approximation," *Fifth International Conference in Computer Vision*, June 20-23, 1995, 902-907.