THE ESSENTIAL SPECTRUM OF PERIODICALLY STATIONARY PULSES IN LUMPED MODELS OF SHORT-PULSE FIBER LASERS*

VRUSHALY SHINGLOT† AND JOHN ZWECK†

Abstract. In modern short pulse fiber lasers there is significant pulse breathing over each round trip of the laser loop. Consequently, averaged models cannot be used for quantitative modeling and design. Instead, lumped models, which are obtained by concatenating models for the various components of the laser, are required. Since the pulses in lumped models are periodic rather than stationary, their linear stability is evaluated with the aid of the monodromy operator obtained by linearizing the round trip operator about the periodic pulse. Conditions are given on the smoothness and decay of the periodic pulse which ensure that the monodromy operator exists on an appropriate Lebesgue function space. A formula for the essential spectrum of the monodromy operator is given which can be used to quantify the growth rate of continuous wave perturbations. This formula is established by showing that the essential spectrum of the monodromy operator equals that of an associated asymptotic operator. Since the asymptotic monodromy operator acts as a multiplication operator in the Fourier domain, it is possible to derive a formula for its spectrum. Although the main results are stated for a particular experimental stretched pulse laser, the analysis shows that they can be readily adapted to a wide range of lumped laser models.

Key words. essential spectrum, evolution semigroups, fiber lasers, monodromy operator, nonlinear optics

AMS subject classifications. 35B10, 35Q56, 37L15, 47D06, 78A60

1. Introduction.

The purpose of this paper is to establish a formula for the essential spectrum of the monodromy operator for a periodic pulse in a lumped model of an experimental short pulse fiber laser. The physical importance of the essential spectrum is that it quantifies the growth rate of continuous wave perturbations seeded by quantum mechanical noise in the system. Such perturbations can have a major impact on the performance of laser-based systems. Since the advent of the soliton laser [26], researchers have invented several generations of short pulse fiber lasers for a variety of applications, including stretched-pulse (dispersion-managed) lasers [22, 32], similariton lasers [7, 11], and the Mamyshev oscillator [28, 31, 33]. The pulses in these lasers typically have durations on the order of 100 fs, peak powers on the order of 1-2 MW, and energy in the 1-50 nJ range. Applications of femtosecond laser technology include frequency comb generation, highly accurate measurement of time, frequency, and distance, optical waveform generation, trace-gas sensing, the search for exoplanets, and laser surgery [3, 8].

Traditionally, mathematical modeling and analysis of short pulse lasers has been based on averaged models, in which each of the physical effects that act on the light pulse is averaged over one round trip of the laser loop to obtain a partial differential equation such as the cubic-quintic complex Ginzburg-Landau equation (CQ-CGLE) or the Haus master equation (see [23] for a review). This approach has been successfully applied to soliton lasers for which the pulse maintains its shape as it propagates over each round trip. In particular, analytical and computational methods have been developed to find stationary pulse solutions of these equations and to analyze their stability using soliton perturbation theory [12, 13, 15, 21, 25]. However, as is highlighted in the survey paper of Turitsyn et al. [35], averaged models cannot be used

*Submitted to the editors 6/22/2022.

Funding: This work was funded by the NSF under DMS-2106203.

†Department of Mathematical Sciences, The University of Texas at Dallas, Richardson, TX 75080, USA (vrushaly.shinglot@utdallas.edu, zweck@utdallas.edu, https://personal.utdallas.edu/~zweck/).

for the quantitative modeling and design of modern short pulse lasers since from one generation of laser to the next there has been a dramatic increase in the amount by which the pulse varies over each round trip.

Instead, the computational modeling of modern short pulse lasers should be based on lumped models obtained by concatenating models for the various components of the laser. Typically short pulse lasers include an optical fiber amplifier, segments of single-mode fiber, a saturable absorber, a dispersion compensating element, a spectral filter, and an output coupler. Different laser designs are characterized by different orderings of the components around the loop and by different sets of physical parameters for each component. Depending on the modeling goal, the models of the individual components may be phenomenological or derived from physical laws. With a lumped model, the pulse changes shape as it propagates through the various components of the laser system, returning to the same shape once per round trip. We call such pulses *periodically stationary* to distinguish them from the stationary pulses in a soliton laser.

The key goals for the modeling of short pulse lasers are to find parameter regions in which stationary or periodically stationary solutions exist, determine the stability of these pulses, and within the stability region to optimize the pulse parameters and noise performance for specific applications.

Building on analytical work of Kaup [21] and Haus [12, 13], Menyuk [25] developed a computational approach to the modeling of stationary pulse solutions of averaged models. With this method, stationary pulses are found using a root finding method and their linear stability is determined by computing the spectrum of the linearization of the governing equation about the pulse. (We recall that the spectrum of an operator on a function space is the union of the essential spectrum and the eigenvalues). In this context the essential spectrum is elementary to calculate with the aid of Weyl's essential spectrum theorem [17]. While Menyuk computes the eigenvalues by solving a nonlinear eigenproblem involving a matrix discretization of the differential operator [29, 36], analytical and computational Evans function methods have also been developed for the CQ-CGLE and for nonlocal equations such as the Haus master equation [16, 18, 19].

Extending this approach to periodically stationary pulses in lumped laser models is significantly more challenging. In [30], building on a method of Ambrose and Wilkening for computing periodic solutions of partial differential equations [2], we developed an optimization method to find periodically stationary pulses. Each iteration of the optimization algorithm involves solving the equations in the model over one round trip of the laser. In analogy with the Floquet theory of periodic solutions of ordinary differential equations [34], we expect that the linear stability of the resulting periodic pulse will be determined by the spectrum of the monodromy operator of the linearization of the lumped model about the pulse. Indeed, it should be possible to rigorously establish such a result by generalizing the Floquet stability theory for parabolic partial differential equations developed by Lunardi [24]. In [30] we also presented a formula for the essential spectrum of the monodromy operator and obtained excellent agreement between the formula and a subset of the eigenvalues of a matrix discretization of the operator. This agreement was shown for a lumped model of an experimental stretched pulse laser of Kim et al [22]. The purpose of the current paper is to prove the essential spectrum formula announced in [30]. Our approach builds upon that in Zweck et al. [39] which dealt with the simpler case of periodically stationary pulse solutions of the constant-coefficient CQ-CGLE.

Since we do not yet know how to formulate conditions to ensure that there exists a periodically stationary pulse solution to the lumped model, for the results in this paper we simply assume that the parameters in the model have been chosen so that such a pulse exists. This assumption is reasonable since we have solid numerical evidence for the existence of such pulses [30]. The first main result of the paper, Theorem 4.4, provides conditions on the regularity and decay of the pulse which guarantee that the monodromy operator exists on an appropriate L^2-function space. Since it is not possible to calculate the essential spectrum of the monodromy operator directly, we instead compute the essential spectrum of an associated asymptotic monodromy operator. The asymptotic operator is defined by taking the limit as the spatial variable goes to infinity of the monodromy operator. Intuitively, the spectrum of the asymptotic operator provides information about the growth rate of noise perturbations far from the pulse. The second main result, Theorem 4.6, is a formula for the essential spectrum of the asymptotic monodromy operator. This result is established in the Fourier domain, where the asymptotic operator acts as a multiplication operator on a space of $\mathbb{C}^2$-valued functions. The proof relies on a general formula we derive for the spectrum of a multiplication operator on $L^2(\mathbb{R}, \mathbb{C}^2)$. The proof of this general formula builds on a similar well known formula in the case of scalar-valued functions [5], but the case of vector-valued functions involves some additional technicalities. Finally, in the third main result, Theorem 4.7, we establish conditions which guarantee that the essential spectrum of the monodromy operator equals that of the asymptotic operator.

To keep the presentation as concrete as possible, rather than attempting to formulate an abstract definition of a general lumped model of a short pulse laser, the theorems are formulated and proved for the Kim laser we modeled in [30]. However, based on the discussion at the beginning of this introduction, we anticipate that the results can easily be adapted to most lumped laser models. In particular, the formula we derive for the essential spectrum is independent of the order of the components in the model. Furthermore, provided that the conditions in the remark following Theorem 4.7 still hold, the models for the components can be switched out for other models, and additional components such as a spectral filter can be added. Finally, the conditions on the physical parameters we impose in the main results hold generically.

From a technical point of view there are two main challenges in extending the results on the constant coefficient CQ-CGLE in [39] to lumped laser models. The first challenge is that nonlocality of the gain saturation in the Haus master equation complicates the proofs of the main theorems. The physical implications of the nonlocality of the gain saturation are discussed in Section 5. The second challenge is that the monodromy operator is defined as a composition of solution operators for each component of the model, which requires adopting a different point of view, especially in the proof of the third main result. The combination of these two challenges ultimately means that the formula for the essential spectrum in the lumped model has a different character from the CQ-CGLE case.

The paper can be outlined as follows. In Section 2, we describe the lumped model of the experimental stretched pulse laser of Kim et al. [22] and define the round trip operator, $\mathcal{R}$. In Section 3, we linearize $\mathcal{R}$ about a periodically stationary pulse, ψ, to obtain the monodromy operator, $\mathcal{M}$, and the associated asymptotic monodromy operator, $\mathcal{M}_\infty$. In Section 4, we state the three main theorems of the paper, including formulating the hypotheses on ψ we need to obtain these results. We also state the formula we derived for the essential spectrum of $\mathcal{M}$. In Section 5 we present some simulation results based on this formula. In Section 6, we prove the first main theorem on the existence and regularity properties of $\mathcal{M}$. This proof relies on the concept of an evolution system in semigroup theory [27] in which linear partial differential equations of the form $\partial_t \mathbf{u} = \mathcal{L}(t)\mathbf{u}$ are regarded as ordinary differential equations for trajectories,

$t \mapsto \mathbf{u}(t)$, in an infinite dimensional Banach space. The estimates in the proof of the technical Lemma 6.7 are relegated to Appendix A. In Section 7, we derive a formula for the spectrum of a general multiplication operator on $L^2(\mathbb{R}, \mathbb{C}^2)$, and in Section 8 we apply this formula to calculate the essential spectrum of $\mathcal{M}_\infty$. In Sections 9 and 10, we prove two theorems concerning the linearized differential operator, $\mathcal{L}(t)$, in the fiber amplifier and its asymptotic counterpart, $\mathcal{L}_\infty(t)$. The first result states that $\mathcal{L}(t)$ is a relatively compact perturbation of $\mathcal{L}_\infty(t)$ and the second result states that the semigroup of the operator $\mathcal{L}_\infty(t)$ is analytic. Finally, these results are used in Section 11 to prove the third main theorem that the essential spectrum of $\mathcal{M}$ equals the essential spectrum of $\mathcal{M}_\infty$.

2. Mathematical Model. In the left panel of Fiigure 1, we show a system diagram for the lumped model of the stretched pulse laser of Kim et al. [22]. A light pulse circulates around the loop, passing through a saturable absorber (SA), a segment of single mode fiber (SMF1), a fiber amplifier (FA), a second segment of single mode fiber (SMF2), a dispersion compensation element (DCF), and an output coupler (OC). After several round trips, the light circulating in the loop forms into a pulse that changes shape as it propagates through the different components, returning to the same shape each time it returns to the same position in the loop. In the right panel of Figure 1 we show the profile of such a periodically stationary pulse at the output of each component. The goal of this paper is to study the spectral stability of periodically stationary pulses in lumped models of fiber lasers.

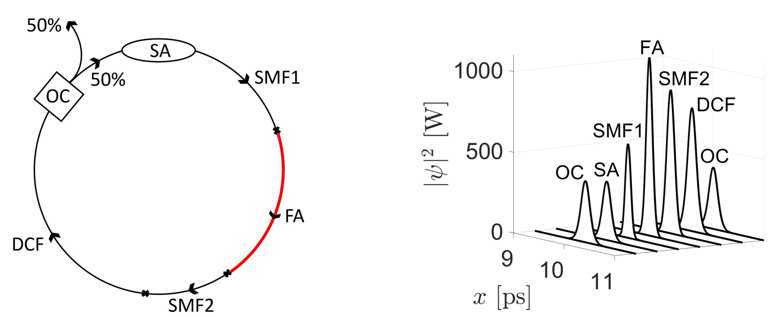

FIG. 1. *Left: System diagram of the stretched pulse laser of Kim et al. [22]. Right: Instantaneous power of the periodically stationary pulse exiting each component of the laser.*

At each position in the loop, we model the complex electric field envelope of the light as a function, $\psi = \psi(x)$, of a spatial variable, x, across the pulse. The pulse is normalized so that $|\psi(x)|^2$ is the instantaneous power. We assume that the function, ψ, is an element of the Lebesgue space, $L^2(\mathbb{R}, \mathbb{C})$, of square integrable, complex-valued functions on $\mathbb{R}$. We model each component of the laser as a transfer function, $\mathcal{P} : L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$, so that

$$\psi_{\text{out}} = \mathcal{P}\psi_{\text{in}}, \tag{2.1}$$

where ψ_{in} and ψ_{out} are the pulses entering and exiting the component. The components in the model come in two flavors: discrete and continuous. By a discrete component we mean one in which the action of the operator, $\mathcal{P}$, on the input pulse, ψ_{in}, is essentially obtained in one step, for example by the application of an explicit formula. In our model of the Kim laser, the discrete components are the saturable absorber, dispersion compensation element, and output coupler. Short-pulse fiber lasers sometimes also include a spectral filter that is modeled as a discrete component. By a continuous component, we mean one in which the action of the operator, $\mathcal{P}$, on the input pulse, ψ_{in}, is modeled by solving a nonlinear wave equation with initial condition, ψ_{in}, from the input to the output of the component. In fiber lasers the continuous components are those that involve the propagation of a light pulse through a segment of nonlinear optical fiber. For our model of the Kim laser these are the fiber amplifier and the two segments of single mode fiber. Note that we have chosen to model the dispersion compensation element as a discrete component, since it is modeled by a constant-coefficient linear partial differential equation which has an analytical solution in the Fourier domain.

With a lumped model, the propagation of a light pulse once around the laser loop is modeled by the round trip operator, $\mathcal{R} : L^2(\mathbb{R}, \mathbb{C}) \to L^2(\mathbb{R}, \mathbb{C})$, which is given by the composition of the transfer functions of all the components. For our model of the Kim laser, the round trip operator is given by

$$\mathcal{R} = \mathcal{P}^{\text{OC}} \circ \mathcal{P}^{\text{DCF}} \circ \mathcal{P}^{\text{SMF2}} \circ \mathcal{P}^{\text{FA}} \circ \mathcal{P}^{\text{SMF1}} \circ \mathcal{P}^{\text{SA}}. \tag{2.2}$$

We say that $\psi_0 \in L^2(\mathbb{R}, \mathbb{C})$ is a *periodically stationary pulse* if

$$\mathcal{R}(\psi_0) = e^{i\theta}\psi_0, \tag{2.3}$$

for some constant phase, $\theta \in [0, 2\pi)$. For the Kim laser, ψ_0 is the pulse at the input to the saturable absorber. For each component, we let ψ_{in} denote the pulse obtained by propagating the periodically stationary pulse, ψ_0, from the input to the SA to the input to that component. For the continuous fiber components we let ψ denote the pulse propagating through that fiber. In [30], we formulated the problem of discovering periodically stationary pulses as that of finding a zero of the Poincaré map functional, $\mathcal{E} : L^2(\mathbb{R}, \mathbb{C}) \times [0, 2\pi) \to \mathbb{R}$, given by

$$\mathcal{E}(\psi_0, \theta) = \frac{1}{2} \left\| \mathcal{R}(\psi_0) - e^{i\theta}\psi_0 \right\|^2_{L^2(\mathbb{R},\mathbb{C})}. \tag{2.4}$$

Since $\mathcal{E} \geq 0$, in practice we minimize $\mathcal{E}$ with respect to ψ_0 and θ using a gradient-based iterative optimization method. In the right panel of Figure 1, we plot the optical power of a periodically stationary pulse obtained using this method.

We now describe the model for the propagation of a light pulse, $\psi = \psi(t, x)$, through the fiber amplifier. Here t denotes position along the fiber, with $0 \leq t \leq L_{\text{FA}}$, where L_{FA} is the length of the fiber amplifier. We note that t is a local evolution variable that is only defined within the fiber amplifier. Mathematically, we regard x as the spatial variable across the pulse. Physically speaking, it is a fast time variable defined relative to a frame moving at the group velocity [38]. Our model for propagation in the fiber amplifier is based on the Haus master equation [12], which is a generalization of the nonlinear Schródinger equation that includes gain that saturates at high energy and is of finite bandwidth. Specifically, we model the transfer function, $\mathcal{P}^{\text{FA}}$, of a fiber amplifier of length, L_{FA}, as $\psi_{\text{out}} = \mathcal{P}^{\text{FA}}\psi_{\text{in}}$, where $\psi_{\text{out}} = \psi(L_{\text{FA}}, \cdot)$ is obtained by solving the initial value problem

$$\begin{aligned} &\partial_t \psi = \left[\frac{g(\psi)}{2} \left(1 + \frac{1}{\Omega_g^2} \partial_x^2 \right) - \tfrac{i}{2}\beta_{\text{FA}} \partial_x^2 + i\gamma|\psi|^2 \right] \psi, \qquad \text{for } 0 \leq t \leq L_{\text{FA}}, \\ &\psi(0, \cdot) = \psi_{\text{in}}. \end{aligned} \tag{2.5}$$

Here, $g(\psi)$ is the saturable gain given by

$$g(\psi) = \frac{g_0}{1 + E(\psi)/E_{\text{sat}}}, \tag{2.6}$$

where g_0 is the unsaturated gain, $E_{\rm sat}$ is the saturation energy, and $E(\psi)$ is the pulse energy, which is given by

$$E(\psi) = \int_{\mathbb{R}} |\psi(\cdot, x)|^2 dx. \tag{2.7}$$

We note that the energy, and hence the saturable gain, are nonlocal in the spatial variable, x, and that they depend on the evolution variable, t, since ψ does. The finite bandwidth of the amplifier is modeled using a Gaussian filter with bandwidth, Ω_g. In (2.5), $\beta_{\rm FA}$ is the chromatic dispersion coefficient and γ is the nonlinear Kerr coefficient.

Similarly, we model the transfer function, $\mathcal{P}^{\rm SMF}$, of a segment of single mode fiber of length, $L_{\rm SMF}$, as $\psi_{\rm out} = \mathcal{P}^{\rm SMF}\psi_{\rm in}$, where $\psi_{\rm out} = \psi(L_{\rm SMF}, \cdot)$ is obtained by solving the initial value problem for the nonlinear Schrödinger equation given by

$$\begin{aligned} &\partial_t \psi = -\tfrac{i}{2}\beta_{\rm SMF}\partial_x^2\psi + i\gamma|\psi|^2\psi, \qquad \text{for } 0 \le t \le L_{\rm SMF}, \\ &\psi(0, \cdot) = \psi_{\rm in}. \end{aligned} \tag{2.8}$$

We model the dispersion compensation element as $\mathcal{P}_{\rm DCF} = \mathcal{F}^{-1} \circ \widehat{\mathcal{P}}^{\rm DCF} \circ \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform and

$$\widehat{\psi}_{\rm out}(\omega) = (\widehat{\mathcal{P}}^{\rm DCF}\widehat{\psi}_{\rm in})(\omega) = \exp\left(i\omega^2\beta_{\rm DCF}/2\right)\widehat{\psi}_{\rm in}(\omega), \tag{2.9}$$

with $\widehat{\psi} = \mathcal{F}(\psi)$. We observe that (2.9) is the solution to the initial value problem for the linear equation obtained by setting $\gamma = 0$, $\beta_{\rm SMF} = \beta_{\rm DCF}$ and $L_{\rm SMF} = 1$ in (2.8).

We model the saturable absorber using the fast saturable loss transfer function [37], $\mathcal{P}^{\rm SA}$, given by

$$\psi_{\rm out} = \mathcal{P}^{\rm SA}(\psi_{\rm in}) = \left(1 - \frac{\ell_0}{1 + |\psi_{\rm in}|^2/P_{\rm sat}}\right)\psi_{\rm in}, \tag{2.10}$$

where ℓ_0 is the unsaturated loss and $P_{\rm sat}$ is the saturation power. With this model, $\psi_{\rm out}$ at x only depends on $\psi_{\rm in}$ at the same value of x. Finally, we model the transfer function, $\mathcal{P}^{\rm OC}$, of the output coupler as

$$\psi_{\rm out} = \mathcal{P}^{\rm OC}\psi_{\rm in} = \ell_{\rm OC}\,\psi_{\rm in}, \tag{2.11}$$

where $(\ell_{\rm OC})^2$ is the power loss at the output coupler.

3. Linearization of the Round Trip Operator.

In this section, we derive the linearizations, $\mathcal{U}$, about a pulse of each of the operators, $\mathcal{P}$, defined in Section 2. By the chain rule, the linearization, $\mathcal{M}$, of the round trip operator, $\mathcal{R}$, about a periodically stationary pulse, ψ_0, is equal to the composition of the linearized transfer functions, $\mathcal{U}$, i.e.,

$$\mathcal{M} = \mathcal{U}^{\rm OC} \circ \mathcal{U}^{\rm DCF} \circ \mathcal{U}^{\rm SMF2} \circ \mathcal{U}^{\rm FA} \circ \mathcal{U}^{\rm SMF1} \circ \mathcal{U}^{\rm SA}. \tag{3.1}$$

In analogy with the monodromy matrix in the Floquet theory of periodic solutions of ODE's [34], we call $\mathcal{M}$ the *monodromy operator* of the linearization of the round trip operator, $\mathcal{R}$, about the periodically stationary pulse, ψ_0.

Because the linearization of the partial differential equations in the model involves the complex conjugate of ψ, we reformulate the model as a system of equations for the column vector $\boldsymbol{\psi} = [\mathrm{Re}(\psi), \mathrm{Im}(\psi)]^T \in \mathbb{R}^2$. For example, the transfer function of the fiber amplifier is reformulated as the operator, $\mathcal{P}^{\mathrm{FA}} : L^2(\mathbb{R},\mathbb{R}^2) \to L^2(\mathbb{R},\mathbb{R}^2)$, given by $\boldsymbol{\psi}_{\mathrm{out}} = \mathcal{P}^{\mathrm{FA}}\boldsymbol{\psi}_{\mathrm{in}}$, where $\boldsymbol{\psi}_{\mathrm{out}} = \boldsymbol{\psi}(\mathrm{L}_{\mathrm{FA}},\cdot)$ is obtained by solving the initial value problem

$$\begin{aligned} \partial_t \boldsymbol{\psi} &= \left[\tfrac{g(\boldsymbol{\psi})}{2}\left(1+\tfrac{1}{\Omega_g^2}\partial_x^2\right) - \tfrac{\beta}{2}\mathbf{J}\partial_x^2 + \gamma\left\|\boldsymbol{\psi}\right\|^2\mathbf{J}\right]\boldsymbol{\psi}, \\ \boldsymbol{\psi}(0,\cdot) &= \boldsymbol{\psi}_{\mathrm{in}}, \end{aligned} \tag{3.2}$$

where $\mathbf{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, and $\|\cdot\|$ is the standard Euclidean norm on $\mathbb{R}^2$.

The linearized transfer function, $\mathcal{U}^{\mathrm{FA}} : L^2(\mathbb{R},\mathbb{R}^2) \to L^2(\mathbb{R},\mathbb{R}^2)$, in the fiber amplifier is given by $\boldsymbol{u}_{\mathrm{out}} = \mathcal{U}^{\mathrm{FA}}\boldsymbol{u}_{\mathrm{in}}$, where $\boldsymbol{u}_{\mathrm{out}} = \boldsymbol{u}(L_{\mathrm{FA}},\cdot)$ is obtained by solving the linearized initial value problem

$$\begin{aligned} \partial_t \boldsymbol{u} &= \left[g(\boldsymbol{\psi})\mathbf{K} + \mathbf{L} + \mathbf{M}_1(\boldsymbol{\psi}) + \mathbf{M}_2(\boldsymbol{\psi})\right]\boldsymbol{u} + \mathbf{P}(\boldsymbol{\psi},\boldsymbol{u}), \qquad \text{for } 0 \le t \le L_{\mathrm{FA}} \\ \boldsymbol{u}(0,\cdot) &= \boldsymbol{u}_{\mathrm{in}}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} \mathbf{K} &= \tfrac{1}{2}\left(1+\tfrac{1}{\Omega_g^2}\partial_x^2\right), & \mathbf{L} &= -\tfrac{\beta}{2}\mathbf{J}\partial_x^2, \\ \mathbf{M}_1(\boldsymbol{\psi}) &= \gamma\left\|\boldsymbol{\psi}\right\|^2\mathbf{J}, & \mathbf{M}_2(\boldsymbol{\psi}) &= 2\gamma\mathbf{J}\boldsymbol{\psi}\boldsymbol{\psi}^T, \end{aligned} \tag{3.4}$$

and

$$\mathbf{P}(\boldsymbol{\psi},\boldsymbol{u}) = -\tfrac{g^2(\boldsymbol{\psi})}{g_0 E_{\mathrm{sat}}}\left[\left(1+\tfrac{1}{\Omega_g^2}\partial_x^2\right)\boldsymbol{\psi}\right]\int_{-\infty}^{\infty}\boldsymbol{\psi}^T(x)\boldsymbol{u}(x)dx \tag{3.5}$$

is a nonlocal operator. The non-locality of $\mathbf{P}$, which arises because the gain saturation depends on the total energy of the pulse, makes the analysis more challenging for the fiber amplifier than for a segment of single mode fiber. The linearized transfer function, $\mathcal{U}^{\mathrm{SMF}}$, of a segment of single mode fiber is obtained by setting $g(\boldsymbol{\psi}) = 0$ in (3.3) and (3.5).

The linearized transfer function, $\mathcal{U}^{\mathrm{SA}}$, for the saturable absorber is given by

$$\boldsymbol{u}_{\mathrm{out}} = \mathcal{U}^{\mathrm{SA}}(\boldsymbol{\psi}_{\mathrm{in}})\boldsymbol{u}_{\mathrm{in}} = \left(1 - \ell(\boldsymbol{\psi}_{\mathrm{in}}) - \frac{2\ell^2(\boldsymbol{\psi}_{\mathrm{in}})}{\ell_0 P_{\mathrm{sat}}}\boldsymbol{\psi}_{\mathrm{in}}\boldsymbol{\psi}_{\mathrm{in}}^T\right)\boldsymbol{u}_{\mathrm{in}}, \tag{3.6}$$

where

$$\ell(\boldsymbol{\psi}_{\mathrm{in}}) = \frac{\ell_0}{1 + \left\|\boldsymbol{\psi}_{\mathrm{in}}\right\|^2/P_{\mathrm{sat}}}. \tag{3.7}$$

The remaining components, i.e. dispersion compensation fiber and output coupler, already have linear transfer functions, and so $\mathcal{U}^{\mathrm{DCF}} = \mathcal{P}^{\mathrm{DCF}}$ and $\mathcal{U}^{\mathrm{OC}} = \mathcal{P}^{\mathrm{OC}}$.

Because eigenvalues and eigenfunctions can be complex valued, we extend the linearized system to act on complex-valued functions, $\boldsymbol{u} \in L^2(\mathbb{R},\mathbb{C}^2)$, where

$$L^2(\mathbb{R},\mathbb{C}^2) = \{\boldsymbol{u} = \boldsymbol{v} + i\boldsymbol{w} \;:\; \boldsymbol{v},\boldsymbol{w} \in L^2(\mathbb{R},\mathbb{R}^2)\}, \tag{3.8}$$

is the space of $\mathbb{C}^2$-valued functions on $\mathbb{R}$ with the standard Hermitian inner product. Let $\mathcal{T}$ be an operator that acts on $\mathbb{R}^2$-valued functions. We extend $\mathcal{T}$ to act on $\mathbb{C}^2$-valued functions by defining $\mathcal{T}\boldsymbol{u} = \mathcal{T}\boldsymbol{u}_1 + i\mathcal{T}\boldsymbol{u}_2$. where $\boldsymbol{u} = \boldsymbol{u}_1 + i\boldsymbol{u}_2$ with $\boldsymbol{u}_1, \boldsymbol{u}_2 \in L^2(\mathbb{R},\mathbb{R}^2)$. Note that the formulae above for the action of the differential operators and transfer functions on $\mathbb{C}^2$-valued functions, $\boldsymbol{u}$, are the same as for their action on $\mathbb{R}^2$-valued functions, since in both cases we only require $\boldsymbol{\psi}$ to be $\mathbb{R}^2$-valued. The only difference is our interpretation of the function spaces on which they act.

The linear stability of the pulse ψ is determined by the spectrum of the monodromy operator, $\mathcal{M}$, which is the union of the essential spectrum of $\mathcal{M}$ and the eigenvalues of $\mathcal{M}$. In Section 4, we show that the essential spectrum of the monodromy operator is equal to the essential spectrum of an associated *asymptotic monodromy operator*, $\mathcal{M}_\infty$, which is defined by

$$\mathcal{M}_\infty = \mathcal{U}_\infty^{\mathrm{OC}} \circ \mathcal{U}_\infty^{\mathrm{DCF}} \circ \mathcal{U}_\infty^{\mathrm{SMF2}} \circ \mathcal{U}_\infty^{\mathrm{FA}} \circ \mathcal{U}_\infty^{\mathrm{SMF1}} \circ \mathcal{U}_\infty^{\mathrm{SA}}, \tag{3.9}$$

where each operator, $\mathcal{U}_\infty$, is the x-independent operator obtained by taking the limit as $|x| \to \infty$ of the corresponding operator, $\mathcal{U}$. In Section 4, we will impose conditions on the pulse that ensure that these limits exist. Under these conditions, each operator $\mathcal{U}_\infty$ can be obtained by setting $\boldsymbol{\psi} = 0$ in the corresponding formula for $\mathcal{U}$. We refer to the operators, $\mathcal{U}_\infty$, as *asymptotic linearized transfer functions.*

4. Main Results.

In this section, we first state a theorem that establishes the existence, uniqueness, and regularity properties of the monodromy operator, $\mathcal{M}$, given by (3.1). Essentially the same result also holds for the asymptotic monodromy operator, $\mathcal{M}_\infty$, given by (3.9). Then we provide an explicit formula for the essential spectrum of $\mathcal{M}_\infty$. The last major result is a theorem stating that essential spectrum of $\mathcal{M}$ equals that of $\mathcal{M}_\infty$.

Rigorously proving the existence, uniqueness, and regularity of periodically stationary pulse solutions, ψ, of the lumped model is challenging. Instead, for the results in this paper, we assume that a periodically stationary pulse, ψ, exists. This assumption is reasonable since we have strong numerical evidence for the existence of such pulses [30]. We do however need to impose some regularity and decay hypothesis on ψ to guarantee the existence of a monodromy operator for ψ and to prove the results about the essential spectrum. These can be stated as follows.

Hypothesis 4.1. The pulse, ψ_{in}, about which the transfer function, (2.10), of the saturable absorber is linearized has the property that ψ_{in}, $\partial_x \psi_{\mathrm{in}}$, and $\partial_x^2 \psi_{\mathrm{in}}$ are bounded and continuous on $\mathbb{R}$, and ψ_{in} decays exponentially to zero as $x \to \pm\infty$.

Hypothesis 4.2. The pulse, ψ, about which equation (2.8) for each single mode fiber of length, L_{SMF}, is linearized has the following properties:

(a) ψ, $\partial_t \psi$ are continuous in t, uniformly in x;
(b) For each t, the function $\psi(t, \cdot) \in L^\infty(\mathbb{R}, \mathbb{C})$;
(c) For each t, the weak derivative $\partial_x \psi(t, \cdot) \in L^\infty(\mathbb{R}, \mathbb{C})$;
(d) There exist constant $r > 0$ so that

$$\lim_{x \to \pm\infty} e^{r|x|} |\psi(t, x)| = 0, \quad \text{for all } t \in [0, L_{\mathrm{SMF}}]. \tag{4.1}$$

Hypothesis 4.3. In the fiber amplifier of length, L_{FA}, the pulse, ψ, about which (2.5) is linearized has the same properties as in Hypothesis 4.2, in addition to which

(a) For almost all $x \in \mathbb{R}$, ψ is C^2 in t;
(b) For almost all $x \in \mathbb{R}$, $\partial_x^2 \psi$, $\partial_t(\partial_x \psi)$, $\partial_t(\partial_x^2 \psi)$ are continuous in t;
(c) There exists $h \in L^2(\mathbb{R}, \mathbb{R}) \cap L^\infty(\mathbb{R}, \mathbb{R})$ so that

$$\left| \partial_t^{(k)} \partial_x^{(\ell)} \psi(t, x) \right| \le h(x) \quad \text{for } k = 0, 1, \ \ell = 0, 1, 2, \tag{4.2}$$

and

$$\left|\partial_t^2 \psi(t,x)\right| \le h(x), \tag{4.3}$$

for all $t \in [0, L_{\mathrm{FA}}]$ and almost all $x \in \mathbb{R}$.

REMARK. *Property (c) of Hypothesis 4.3 holds if all the functions $\partial_t^{(k)}\partial_x^{(\ell)}\psi$ are bounded and decay exponentially as in property (d) of Hypothesis 4.2.*

Let $\mathcal{B}(X)$ denote the space of bounded linear operators on a Banach space, X. Then we have the following theorem on the existence, unqiueness, and regularity of the monodromy operator.

THEOREM 4.4. *Let $\boldsymbol{\psi}_0$ be a periodically stationary solution of the lumped laser model, i.e., a solution of (2.2). Under Hypotheses 4.1, 4.2, and 4.3, the monodromy operator, $\mathcal{M}$, in (3.1), which is the linearization of the round trip operator, $\mathcal{R}$, about $\boldsymbol{\psi}_0$, has the following properties:*

(a) $\mathcal{M} \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2))$;

(b) $\mathcal{M}(H^2(\mathbb{R}, \mathbb{C}^2)) \subset H^2(\mathbb{R}, \mathbb{C}^2)$;

(c) For each $\boldsymbol{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$, $\boldsymbol{u} = \mathcal{M}(\boldsymbol{v})$ is the unique solution after one round trip of the linearization of $\mathcal{R}$ about $\boldsymbol{\psi}$.

REMARK. *An analgous result holds for the asymptotic monodromy operator, $\mathcal{M}_\infty$, given by (3.9).*

Next, we recall the definition of the essential spectrum used in the results below.

DEFINITION 4.5. *Let $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ be a linear operator with domain, $D(\mathcal{A})$, on a Banach space, X. We suppose that $\mathcal{A}$ is closed and densely defined. The* resolvent set *of $\mathcal{A}$ is*

$$\rho(\mathcal{A}) = \{\lambda \in \mathbb{C} : \mathcal{A} - \lambda \text{ is invertible and } (\mathcal{A} - \lambda)^{-1} \in \mathcal{B}(X)\}, \tag{4.4}$$

and for each $\lambda \in \rho(\mathcal{A})$, the resolvent operator *is $\mathcal{R}(\lambda : \mathcal{A}) = (\mathcal{A} - \lambda)^{-1}$. The* spectrum *of $\mathcal{A}$ is $\sigma(\mathcal{A}) = \mathbb{C} \backslash \rho(\mathcal{A})$. The* point spectrum *of $\mathcal{A}$ is*

$$\sigma_{\mathrm{pt}}(\mathcal{A}) = \{\lambda \in \mathbb{C} : \mathrm{Ker}(\mathcal{A} - \lambda) \neq \{0\}\}. \tag{4.5}$$

The Fredholm point spectrum *of $\mathcal{A}$ is the subset of $\sigma_{\mathrm{pt}}(\mathcal{A})$ defined by*

$$\sigma_{\mathrm{pt}}^{\mathcal{F}}(\mathcal{A}) = \{\lambda \in \mathbb{C} : \mathcal{A} - \lambda \text{ is Fredholm, } \mathrm{Ind}(\mathcal{A} - \lambda) = 0 \text{ and } \mathrm{Ker}(\mathcal{A} - \lambda) \neq \{0\}\}, \tag{4.6}$$

and the essential spectrum *of $\mathcal{A}$ is $\sigma_{\mathrm{ess}}(\mathcal{A}) = \sigma(\mathcal{A}) \backslash \sigma_{\mathrm{pt}}^{\mathcal{F}}(\mathcal{A})$.*

REMARK. *Although $\sigma(\mathcal{A}) = \sigma_{\mathrm{pt}}(\mathcal{A}) \cup \sigma_{\mathrm{ess}}(\mathcal{A})$, this union may not be disjoint.*

REMARK. *There are several inequivalent definitions of the essential spectrum of a closed and densely defined operator. Here, we use the same definition of the essential spectrum as in Zweck et al. [39]. This definition is chosen so that $\sigma_{ess}(\mathcal{A})$ is the largest subset of the spectrum of $\mathcal{A}$ that is invariant under compact perturbations [4].*

Next, we state a formula for the essential spectrum of $\mathcal{M}_\infty$. This formula involves the total dispersion in one round trip of the laser system, which for the stretched pulse laser is given by $\beta_{\mathrm{RT}} = \beta_{\mathrm{SMF1}}\mathrm{L}_{\mathrm{SMF1}} + \beta_{\mathrm{FA}}\mathrm{L}_{\mathrm{FA}} + \beta_{\mathrm{SMF2}}\mathrm{L}_{\mathrm{SMF2}} + \beta_{\mathrm{DCF}}$. Here β_{FA}, β_{SMF}, and β_{DCF}, are the dispersion parameters given in (2.5), (2.8), and (2.9), respectively.

THEOREM 4.6. *Suppose that the hypotheses of Theorem 4.4 hold, and that $\ell_0 \neq 1$ and either (i) $\beta_{RT} \neq 0$ or (ii) $\Omega_g < \infty$ and $\int_0^{L_{\mathrm{FA}}} g(\psi(t))dt \neq 0$. Then the essential spectrum of the asymptotic monodromy operator, $\mathcal{M}_\infty$, in (3.9) is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{M}_\infty) = \sigma(\mathcal{M}_\infty) = \{ \lambda_\pm(\omega) \in \mathbb{C} \mid \omega \in \mathbb{R} \} \cup \{0\}, \tag{4.7}$$

where

$$\lambda_\pm(\omega) = \ell_{\mathrm{OC}}(1-\ell_0) \exp\left\{ \frac{1}{2}\left(1 - \frac{\omega^2}{\Omega_g^2}\right) \int_0^{L_{\mathrm{FA}}} g(\psi(t))dt \right\} \exp\left\{ \pm i \frac{\omega^2}{2} \beta_{RT} \right\}. \tag{4.8}$$

REMARK. *Equation (4.8) can be readily adapted to other lumped fiber laser models, provided that formulae can be found for the Fourier transforms of all the asymptotic linearized transfer functions, $\mathcal{U}_\infty$, in the model. In particular, the formula is independent of the order in which the components are arranged around the loop.*

To prove that the essential spectrum of $\mathcal{M}$ equals that of $\mathcal{M}_\infty$ we require that the linearization of the equation modeling the single mode fiber segments (SMF1 and SMF2) generates an analytic semigroup. To do so, we add an additional spectral filtering term to the nonlinear Schrödinger equation, so that light propagation in these fibers is modeled by

$$\partial_t \psi = -\frac{i}{2}\beta \partial_x^2 \psi + i\gamma|\psi|^2\psi + \epsilon \partial_x^2 \psi, \tag{4.9}$$

where the parameter, ϵ, is required to be positive, but can be arbitrarily small. Provided that $\epsilon > 0$, the semigroup for the linearized equation is analytic (see Section 10). In the frequency domain the additional term corresponds to

$$\partial_t \widehat{\psi}(\omega) = -\epsilon \omega^2 \widehat{\psi}(\omega), \tag{4.10}$$

which models a frequency-dependent loss. The addition of this term is physically reasonable since the loss in optical fiber is wavelength dependent with a minimum at about 1550 nm [1].

THEOREM 4.7. *Suppose that the hypotheses of Theorem 4.4 hold, and that in the fiber amplifier $0 < \Omega_g < \infty$ and $(g_0, \beta) \neq (0,0)$. Furthermore, suppose that the single mode fiber segments are modeled using (4.9) with $\epsilon > 0$. Then the essential spectrum of the monodromy operator, $\mathcal{M}$, in (3.1) is given by*

$$\sigma_{\mathrm{ess}}(\mathcal{M}) \ = \ \sigma_{\mathrm{ess}}(\mathcal{M}_\infty). \tag{4.11}$$

REMARK. *For simplicity we state and prove this theorem for the lumped model of the stretched pulse laser discussed in Section 2. However, (4.11) also holds for a wide range of lumped models of fiber lasers. Specifically, as we will see in the proof, in addition to the hypotheses made about the fiber segments, we just require that the linearizations, $\mathcal{U}$ and $\mathcal{U}_\infty$, of the transfer operators of the input-output devices in the model satisfy*

$$\mathcal{U}, \mathcal{U}_\infty \in \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2)) \cap \mathcal{B}(H^2(\mathbb{R},\mathbb{C}^2)), \tag{4.12}$$

and that an analogue of Theorem 11.2 below holds for each of them.

5. Simulation Results. In this section we use formula (4.8) for the essential spectrum to provide some insights into the roles that the saturable absorber and the saturation of the gain in the fiber amplifier play in stabilizing the periodically stationary pulse circulating in the laser. Further details can be found in [30].

Although we are not modeling it here, in addition to its role in pulse amplification, the fiber amplifier adds spontaneous emission noise to the system [9], which—among other effects such as random timing and phase shifts of the pulse—manifests itself as a random superposition of continuous wave perturbations. If the essential spectrum, $\sigma_{\text{ess}}(\mathcal{M})$, lies inside the unit disc in $\mathbb{C}$, then these continuous wave perturbations decay, which helps ensure pulse stability.

From (2.6) we see that the gain in the fiber amplifier simply depends on the pulse energy. Consequently, each round trip the noise entering the fiber amplifier experiences the same gain as does the pulse. Furthermore, as the pulse propagates through the fiber amplifier, spontaneous emmission noise that is proportional to the gain is added to the system. The saturation of the gain therefore plays a critical role in stabilizing the system, since the gain decreases as the pulse energy increases.

On the other hand, with the model we use for the saturable absorber the response is instantaneous, and is given by

$$\psi_{\text{out}}(x) = \left(1 - \frac{\ell_0}{1 + |\psi_{\text{in}}(x)|^2 / P_{\text{sat}}}\right) \psi_{\text{in}}(x), \tag{5.1}$$

so that the value of the output at x only depends on the input at that x. Therefore, far from the pulse, where $\psi_{\text{in}} \approx 0$, the loss is ℓ_0, whereas in the center of the pulse the loss saturates and is less than ℓ_0. Because the loss saturates at high power, the system can operate so that the gain in the fiber amplifier and the loss in the saturable absorber balance for the pulse, while simultaneously loss exceeds gain far from the pulse. Consequently, noise far from the pulse can be suppressed relative to the peak power of the pulse. The larger ℓ_0 is and/or the smaller P_{sat} is in (5.1), the more the saturable absorber suppresses noise far from the pulse, and the more stable the pulse is to noise perturbations. Already in the 1975, Haus [12] identified the need for a saturable absorber to suppress the growth of continuous waves, while balancing gain and loss for the pulse. Formula (4.8) for the essential spectrum of the monodromy operator quantifies this effect for the first time in a lumped model of a fiber laser.

To ensure that a continuous wave perturbation with frequency ω does not grow, we require that $|\lambda_{\pm}(\omega)| \leq 1$, which, because of the Gaussian factor in (4.8), holds for all ω provided that

$$(\ell_{\text{OC}})^2 (1 - \ell_0)^2 G_{\text{Tot}}^{\text{FA}} \leq 1, \qquad \text{where } G_{\text{Tot}}^{\text{FA}} = \exp\left\{ \int_0^{L_{\text{FA}}} g(\psi(t))\, dt \right\}, \tag{5.2}$$

is approximately equal to the energy gain in the fiber amplifier. That is, far from the pulse the loss experienced by continuous waves must exceed the gain. Although (5.2) looks very simple, the essential spectrum can depend in a complex way on the interplay between all the system parameters, since they all influence the shape of the pulse and hence the total gain, $G_{\text{Tot}}^{\text{FA}}$, in the fiber amplifier.

For the simulation results we present here, we chose the parameters in the model to be similar to those in the experimental stretched pulse laser of Kim [22]. The parameters for the saturable absorber are given below. The saturable absorber is followed by a segment of single mode fiber, SMF1, modeled by (2.8), with $\gamma = 2 \times 10^{-3}$ (Wm)$^{-1}$, $\beta_{\text{SMF1}} = 10$ kfs^2/m, (1 kfs$^2 = 10^{-27}$ s^2), and $L_{\text{SMF1}} = 0.32$ m, a fiber

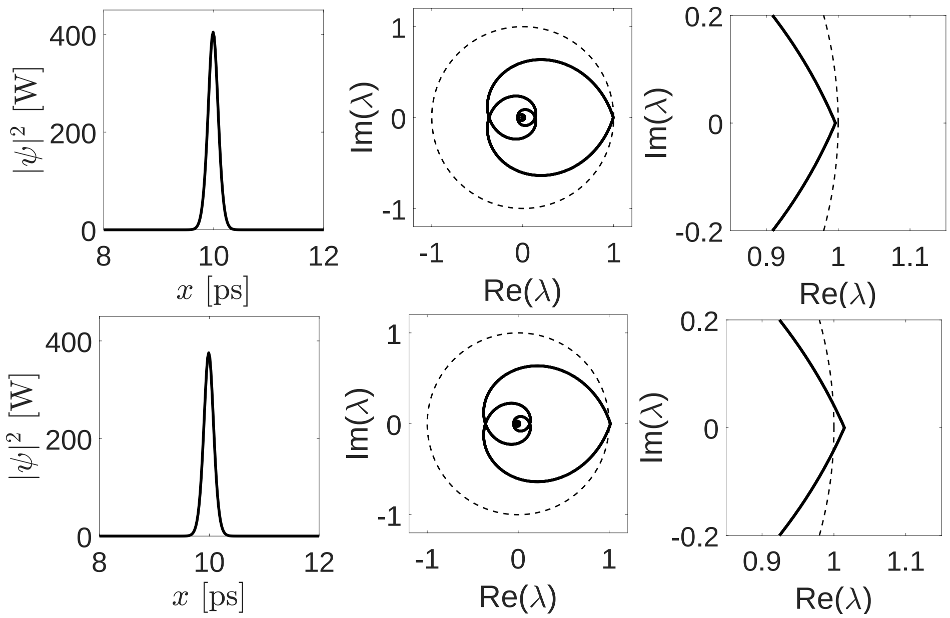

FIG. 2. ***Top row:*** **Left:** *Periodically stationary pulse for* $P_{\text{sat}} = 200$ *W.* **Center and right:** *Essential spectrum,* $\sigma_{\text{ess}}(\mathcal{M})$*, of the monodromy operator associated with the pulse on the left.* ***Bottom row:*** *Corresponding results for* $P_{sat} = 1000$ *W. In both cases,* $\ell_0 = 0.05$*.*

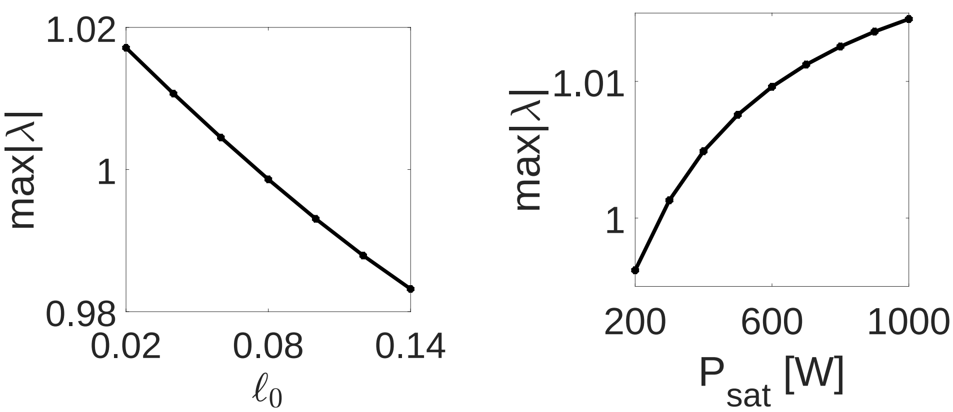

FIG. 3. ***Left:*** *A plot of the maximum real eigenvalue,* $\max|\lambda|$*, vs.* ℓ_0 *when* $P_{sat} = 500$ *W.* ***Right:*** *Corresponding plot in which* P_{sat} *is varied when* $\ell_0 = 0.05$*.*

amplifier, modeled by (2.5), with $g_0 = 6\text{m}^{-1}$, $E_{\text{sat}} = 200$ pJ, $\Omega_g = 50$ THz, $\gamma = 4.4 \times 10^{-3}$ $(\text{Wm})^{-1}$, $\beta_{\text{FA}} = 25$ kfs^2/m, and $L_{\text{FA}} = 0.22$ m, a second segment of single mode fiber, SMF2, with the same parameters as SMF1, but with $L_{\text{SMF2}} = 0.11$ m, a dispersion compensation element with $\beta_{\text{DCF}} = -1$ kfs^2, and a 50% output coupler, modeled by (2.11) with $\ell_{\text{OC}} = \sqrt{0.5}$.

In the top row of Fig. 2, we show the results of simulations performed when $P_{\text{sat}} = 200$ W and $\ell_0 = 0.05$. The pulse, ψ_0, in the left panel was obtained by numerically minimizing the L^2-error between $\mathcal{R}(\psi_0)$ and $e^{i\theta}\psi_0$, over all possible choices of θ [30]. In the center panel we plot the essential spectrum for the pulse in the left panel. We observe that $\sigma_{\text{ess}}(\mathcal{M})$ consists of a pair of counter-rotating spirals whose amplitudes rapidly decay to zero. Since the peak power of the pulse entering the saturable absorber is comparable to P_{sat}, the saturation of the loss is significant, which helps to stabilize the pulse. In the bottom row of Fig. 2, we show the corresponding results with $P_{\text{sat}} = 1000$ W. In this case the saturation of the loss is much weaker, and as we see in the far right panel, there is a range of low frequencies, ω, for which $|\lambda \pm (\omega)| > 1$ and continuous wave perturbations grow.

In the left panel of Fig. 3, we plot the largest value of $|\lambda|$ as a function of ℓ_0 when $P_{\text{sat}} = 500$ W. Since this value remains outside the unit circle as ℓ_0 increases from 0.02 to 0.06, the pulse is unstable over this range. It is only once the unsaturated gain is sufficiently large that condition (5.2) holds and the essential spectrum is stable. Similarly, in the right panel, we show the largest value of $|\lambda|$ as a function of P_{sat} when $\ell_0 = 0.05$. Here, the pulse is unstable for $P_{\text{sat}} > 300$ W, since then the saturation effect is too weak to ensure that the loss experienced by the noise is sufficiently greater than that experienced by the pulse.

6. Existence of the monodromy operator. To prove Theorem 4.4 we use the fact that the monodromy operator, $\mathcal{M}$, is the composition of the linearized transfer functions, $\mathcal{U}$, of each component of the laser. Therefore, we just need to establish the result for each of the operators, $\mathcal{U}$. For the single mode fiber segments and the dispersion compensation element, the result is a special case of the corresponding result for the CQ-CGL equation given in Zweck et al. [39, Theorem 4.1]. For the fast saturable absorber and the fiber amplifier, the results are given in Proposition 6.1 and Theorem 6.4 below.

If X is a Banach space, we let $\|\cdot\|_X$ denote the norm on X. When the context is clear, we sometimes omit the subscript X and simply write $\|\cdot\|$.

PROPOSITION 6.1. *Suppose that Hypothesis 4.1 holds. Then the transfer function, $\mathcal{U}^{\text{SA}}$, given by (3.6) satisfies the first two conclusions of Theorem 4.4.*

Proof. To establish the first conclusion, we use the Cauchy-Schwarz inequality and the fact that $\ell(\boldsymbol{\psi}_{\text{in}}) \leq \ell_0$ (see (3.6)) to obtain

$$\begin{aligned} \|\boldsymbol{u}_{\text{out}}\|_{L^2(\mathbb{R},\mathbb{C}^2)} &\leq (1 + \ell(\boldsymbol{\psi}_{\text{in}}))\, \|\boldsymbol{u}_{\text{in}}\|_{L^2(\mathbb{R},\mathbb{C}^2)} + \frac{2\ell^2(\boldsymbol{\psi}_{\text{in}})}{\ell_0 P_{\text{sat}}} |\boldsymbol{\psi}_{\text{in}}^T \boldsymbol{u}_{\text{in}}|\ \|\boldsymbol{\psi}_{\text{in}}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\ &\leq \left(1 + \ell_0 + \frac{2\ell_0}{P_{\text{sat}}} \|\boldsymbol{\psi}_{\text{in}}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)}\right) \|\boldsymbol{u}_{\text{in}}\|_{L^2(\mathbb{R},\mathbb{C}^2)} . \end{aligned} \tag{6.1}$$

By Hypothesis 4.1, $\boldsymbol{\psi}_{\text{in}} \in L^2(\mathbb{R},\mathbb{C}^2)$. Therefore, $\mathcal{U}^{\text{SA}} \in \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))$. Similarly, to establish the second conclusion, we find that

$$\|\boldsymbol{u}_{\text{out}}\|_{H^2(\mathbb{R},\mathbb{C}^2)} \leq \left(1 + \ell_0 + \frac{2\ell_0}{P_{\text{sat}}} \|\boldsymbol{\psi}_{\text{in}}\|^2_{H^2(\mathbb{R},\mathbb{C}^2)}\right) \|\boldsymbol{u}_{\text{in}}\|_{H^2(\mathbb{R},\mathbb{C}^2)} . \tag{6.2}$$

By Hypothesis 4.1, $\boldsymbol{\psi}_{\text{in}} \in H^2(\mathbb{R},\mathbb{C}^2)$. Therefore, $\mathcal{U}^{\text{SA}} \in \mathcal{B}(H^2(\mathbb{R},\mathbb{C}^2))$. ∎

Next, we establish the existence of an evolution family for the linearization (3.3) of the Haus master equation (2.5), which models propagation in a fiber amplifier of length L_{FA}. Let $t \in [0, L_{\text{FA}}]$ be local time within the fiber amplifier and let $s \in [0, L_{\text{FA}}]$. We study solutions, $\boldsymbol{u} : [s, L_{\text{FA}}] \to H^2(\mathbb{R},\mathbb{C}^2)$, of

$$\begin{aligned} \partial_t \boldsymbol{u} &= \mathcal{L}_{\text{FA}}(t)\boldsymbol{u}, \qquad \text{for } 0 \leq s < t \leq L_{\text{FA}}, \\ \boldsymbol{u}(s) &= \boldsymbol{v}, \end{aligned} \tag{6.3}$$

where $\boldsymbol{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$. Here, $\mathcal{L}_{\text{FA}}(t)$ is the family of operators on $L^2(\mathbb{R}, \mathbb{C}^2)$ given by reformulating (3.3) as

$$\mathcal{L}_{\text{FA}}(t) = \mathbf{B}(t)\partial_x^2 + \widetilde{\mathbf{M}}(t), \tag{6.4}$$

where, setting $g(t) := g(\psi(t))$,

$$\mathbf{B}(t) = \frac{g(t)}{2\Omega_g^2}\mathbf{I} - \frac{\beta}{2}\mathbf{J} \quad \text{and} \quad \widetilde{\mathbf{M}}(t)\boldsymbol{u} = \widetilde{\mathbf{M}}_1(t)\boldsymbol{u} - \boldsymbol{\phi}(t)\langle\boldsymbol{\psi}(t), \boldsymbol{u}\rangle. \tag{6.5}$$

Here, $\langle\,\cdot\,,\,\cdot\,\rangle$ is the L^2-inner product on $L^2(\mathbb{R}, \mathbb{C}^2)$ and

$$\widetilde{\mathbf{M}}_1(t) = \frac{g(t)}{2}\mathbf{I} + \gamma|\boldsymbol{\psi}|^2\mathbf{J} + 2\gamma\mathbf{J}\boldsymbol{\psi}\boldsymbol{\psi}^T \quad \text{and} \quad \boldsymbol{\phi}(t) = \frac{g^2(t)}{g_0 E_{\text{sat}}}\left\{\left(1 + \frac{\partial_x^2}{\Omega_g^2}\right)\boldsymbol{\psi}\right\}. \tag{6.6}$$

Definition 6.2 ([27, 5.5.3]). *A two parameter family of bounded linear operators, $\mathcal{U}(t,s), 0 \le s \le t \le T$, on X is called an* evolution system *if*

(i) $\mathcal{U}(s,s) = \mathcal{I}$, and $\mathcal{U}(t,r) \circ \mathcal{U}(r,s) = \mathcal{U}(t,s)$ for $0 \le s \le r \le t \le T$, and

(ii) $(t,s) \to \mathcal{U}(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

Definition 6.3. *Let $\mathbf{A} = \mathbf{A}(t,x) : [0,\infty) \times \mathbb{R} \to \mathbb{C}^{2\times 2}$ be a bounded matrix-valued function. We define*

$$\|\mathbf{A}\|_\infty = \sup_{(t,x)} \|\mathbf{A}(t,x)\|_{\mathbb{C}^{2\times 2}}\,. \tag{6.7}$$

Theorem 6.4. *Assume that Hypothesis 4.3 holds in the fiber amplifier. Then there exists a unique evolution operator, $\mathcal{U}^{\text{FA}}(t,s) \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2))$, for $0 \le s \le t \le L_{FA}$, where L_{FA} is the length of the fiber amplifier, such that*

1. $\left\|\mathcal{U}^{\text{FA}}(t,s)\right\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \le \exp\left[\left\|\widetilde{\boldsymbol{M}}\right\|_\infty (t-s)\right]$,
2. $\mathcal{U}^{\text{FA}}(t,s)(H^2(\mathbb{R}, \mathbb{C}^2)) \subset H^2(\mathbb{R}, \mathbb{C}^2)$,
3. *For each s, $\mathcal{U}^{\text{FA}}(\cdot, s)$ is strongly continuous in that for all $\boldsymbol{v} \in L^2(\mathbb{R}, \mathbb{C}^2)$, the mapping $t \mapsto \mathcal{U}^{\text{FA}}(t,s)\boldsymbol{v}$ is continuous, and*
4. *For each $\boldsymbol{v} \in H^2(\mathbb{R}, \mathbb{C}^2)$, the function $\boldsymbol{u}(t) = \mathcal{U}^{\text{FA}}(t,s)\boldsymbol{v}$ is the unique solution of the initial value problem (6.3) for which $\boldsymbol{u} \in C([s, L_{\text{FA}}), H^2(\mathbb{R}, \mathbb{C}^2))$ and $\boldsymbol{u} \in C^1((s, L_{\text{FA}}), L^2(\mathbb{R}, \mathbb{C}^2))$.*

Proof. The result follows from [27, Theorems 5.2.3 and 5.4.8]. Lemmas 6.5 to 6.7 below guarantee that the assumptions of these theorems hold. ∎

Lemma 6.5. *The linear operator, $\boldsymbol{B}(t)\partial_x^2 : H^2(\mathbb{R}, \mathbb{C}^2) \subset L^2(\mathbb{R}, \mathbb{C}^2) \to L^2(\mathbb{R}, \mathbb{C}^2)$ is closed with domain $H^2(\mathbb{R}, \mathbb{C}^2)$. Furthermore, $(0,\infty) \subset \rho(\boldsymbol{B}(t)\partial_x^2)$ and the resolvent operator satisfies*

$$\left\|\mathcal{R}(\lambda : \boldsymbol{B}(t)\partial_x^2)\right\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \le \frac{1}{\lambda}, \qquad \textit{for all } \lambda > 0. \tag{6.8}$$

Consequently, $\boldsymbol{B}(t)\partial_x^2$ is the infinitesimal generator of a C_0-semigroup on $L^2(\mathbb{R}, \mathbb{C}^2)$.

Proof. Equation (6.8) follows immediately from [39, Lemma 4.1]. The proof is completed by invoking the Hille-Yosida Theorem [27, 1.3.1]. ∎

Lemma 6.6. *Assume that Hypothesis 4.3 is met. Then there exists $K > 0$ such that for all $t \in [0, L_{\text{FA}}]$*

$$\left\|\widetilde{\boldsymbol{M}}(t)\right\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} < K. \tag{6.9}$$

Proof. We have

$$\left\|\widetilde{\mathbf{M}}(t)\boldsymbol{u}\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \leq \left\|\widetilde{\mathbf{M}}_1\right\|_\infty \|\boldsymbol{u}\|_{L^2(\mathbb{R},\mathbb{C}^2)} + \|\boldsymbol{\phi}(t)\langle\boldsymbol{\psi}(t),\boldsymbol{u}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)}. \tag{6.10}$$

Let $\|\mathbf{A}\|_F$ denote the Frobenius norm of a matrix $\mathbf{A}$. We estimate the first term in (6.10) by

$$\begin{aligned}
\left\|\widetilde{\mathbf{M}}_1\right\|_\infty^2 &\leq \sup_{(t,x)\in[0,L_{\mathrm{FA}}]\times\mathbb{R}} \left\|\widetilde{\mathbf{M}}_1(t,x)\right\|_F^2 \\
&= \sup_{(t,x)\in[0,L_{\mathrm{FA}}]\times\mathbb{R}} \sum_{i,j=1}^{2} \left|\frac{g(t)}{2}\mathbf{I}_{ij} + \gamma|\boldsymbol{\psi}(t,x)|^2\mathbf{J}_{ij} + 2\gamma\left[\mathbf{J}\boldsymbol{\psi}(t,x)\boldsymbol{\psi}^T(t,x)\right]_{ij}\right|^2 \\
&\leq \sup_{(t,x)\in[0,L_{\mathrm{FA}}]\times\mathbb{R}} \left\{\frac{g^2(t)}{4}\sum_{i,j=1}^{2}|I_{ij}|^2 + \gamma^2|\boldsymbol{\psi}(t,x)|^4\sum_{i,j=1}^{2}|J_{ij}|^2\right. \\
&\quad + 4\gamma^2\sum_{i,j=1}^{2}\left|\left[J\boldsymbol{\psi}(t,x)\boldsymbol{\psi}^T(t,x)\right]_{ij}\right|^2 + \gamma g(t)|\boldsymbol{\psi}(t,x)|^2\sum_{i,j=1}^{2}|I_{ij}||J_{ij}| \\
&\quad + 4\gamma^2|\boldsymbol{\psi}(t,x)|^2\sum_{i,j=1}^{2}|J_{ij}|\left|\left[J\boldsymbol{\psi}(t,x)\boldsymbol{\psi}^T(t,x)\right]_{ij}\right| \\
&\quad \left. + 2\gamma g(t)\sum_{i,j=1}^{2}|I_{ij}|\left|\left[J\boldsymbol{\psi}(t,x)\boldsymbol{\psi}^T(t,x)\right]_{ij}\right|\right\} \\
&= \sup_{(t,x)\in[0,L_{\mathrm{FA}}]\times\mathbb{R}} \left\{\frac{g^2(t)}{2} + 10\gamma^2|\boldsymbol{\psi}(t,x)|^4 + 4\gamma g(t)|\operatorname{Re}(\boldsymbol{\psi}(t,x))\operatorname{Im}(\boldsymbol{\psi}(t,x))|\right\} \\
&\leq \frac{g_0^2}{2} + \sup_{(t,x)\in[0,L_{\mathrm{FA}}]\times\mathbb{R}} \left\{10\gamma^2|\boldsymbol{\psi}(t,x)|^4 + 4\gamma g_0|\operatorname{Re}(\boldsymbol{\psi}(t,x))\operatorname{Im}(\boldsymbol{\psi}(t,x))|\right\},
\end{aligned}$$

which is finite by Hypothesis 4.3.

As for the second term in (6.10), by the Cauchy-Schwarz inequality,

$$\begin{aligned}
\|\boldsymbol{\phi}(t)\langle\boldsymbol{\psi}(t),\boldsymbol{u}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)} &\leq \|\boldsymbol{\phi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}\|\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}\|\boldsymbol{u}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\leq \frac{g_0}{E_{\mathrm{sat}}}\left\|\boldsymbol{\psi}(t) + \frac{\partial_x^2\boldsymbol{\psi}(t)}{\Omega_g^2}\right\|_{L^2(\mathbb{R},\mathbb{C}^2)}\|\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}\|\boldsymbol{u}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\leq \max\left\{1,\frac{1}{\Omega_g^2}\right\}\frac{g_0}{E_{\mathrm{sat}}}\|\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}^2\|\boldsymbol{u}\|_{L^2(\mathbb{R},\mathbb{C}^2)}.
\end{aligned}$$

The result now follows, since $\|\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)} < \infty$ by Hypothesis 4.3. ◻

Combining [27, Theorem 5.2.3] and Lemmas 6.5 and 6.6, we conclude that $\{\mathcal{L}_{\mathrm{FA}}(t)\}_{t\in[0,L_{\mathrm{FA}}]}$ is a stable family of infinitesimal generators of C_0-semigroups on $L^2(\mathbb{R},\mathbb{C}^2)$. This is the first assumption in [27, Theorem 5.4.8]. The following Lemma establishes the second assumption.

LEMMA 6.7. *Suppose that Hypothesis 4.3 holds. Then for each* $\boldsymbol{v} \in H^2(\mathbb{R},\mathbb{C}^2)$, *we have that* $F(\cdot) = \mathcal{L}_{\mathrm{FA}}(\cdot)\boldsymbol{v} : (0, L_{\mathrm{FA}}) \to L^2(\mathbb{R},\mathbb{C}^2)$ *is* C^1.

Proof. We show that F is differentiable with $F'(t) = \partial_t\mathcal{L}_{\mathrm{FA}}(t)\boldsymbol{v}$. The proof that F' is continuous is similar. By Hypothesis 4.3, $\mathcal{L}_{\mathrm{FA}}(t)\boldsymbol{v}$, $\partial_t\mathcal{L}_{\mathrm{FA}}(t)\boldsymbol{v} \in L^2(\mathbb{R},\mathbb{C}^2)$. In

Appendix A, we show that

$$
\begin{aligned}
&\|F(t+h)-F(t)-hF'(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}\\
&\le \Big\{2\sqrt{2}hG_1(h)+2\sqrt{2}hG_2(h)+\frac{g_0C}{E_{\text{sat}}}h\,\|\boldsymbol{\psi}(t+h)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\,G_3(h)\\
&\quad+\frac{2g_0^2C}{E_{\text{sat}}^2}h^2\sup_{\tau\in(t,t+h)}|E'(\tau)|\,\|\boldsymbol{\psi}(\tau)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\,\|\partial_t\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\\
&\quad+\frac{2g_0C}{E_{\text{sat}}}h^2\sup_{\tau\in(t,t+h)}\|\partial_t\boldsymbol{\psi}(\tau)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\,\|\partial_t\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\\
&\quad+hG_4(h)\,\|\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\Big\}\,\|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)}\,,
\end{aligned}
\tag{6.11}
$$

where

$$
\begin{aligned}
G_1(h)&=\sup_{\tau\in(t,t+h)}\|(\partial_t\mathbf{B})(\tau)-(\partial_t\mathbf{B})(t)\|_{\mathbb{C}^{2\times2}}\,,\\
G_2(h)&=\sup_{(\tau,x)\in(t,t+h)\times\mathbb{R}}\left\|(\partial_t\widetilde{\mathbf{M}_1})(\tau,x)-(\partial_t\widetilde{\mathbf{M}_1})(t,x)\right\|_{\mathbb{C}^{2\times2}}\,,\\
G_3(h)&=\sup_{\tau\in(t,t+h)}\|(\partial_t\boldsymbol{\psi})(\tau)-(\partial_t\boldsymbol{\psi})(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}\,,\\
G_4(h)&=\sup_{\tau\in(t,t+h)}\|(\partial_t\boldsymbol{\phi})(\tau)-(\partial_t\boldsymbol{\phi})(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}\,.
\end{aligned}
\tag{6.12}
$$

Next, we observe that $\exists C>0$ such that

$$
G_1(h)=C\sup_{\tau\in(t,t+h)}\left|g^2(\tau)E'(\tau)-g^2(t)E'(t)\right|. \tag{6.13}
$$

By Hypothesis 4.3 and the differentiation under the integral sign theorem [14], g and E' are C^1 which implies that $G_1(h)\to0$ as $h\to0$. Also by Hypothesis 4.3, and applying the Lebesgue dominated convergence theorem as needed, we conclude that $G_j(h)\to0$ as $h\to0$ for $j=2,3,4$. Consequently,

$$
\|F(t+h)-F(t)-hF'(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}\le hG(h), \tag{6.14}
$$

where $\lim_{h\to0}G(h)=0$. Hence, F is differentiable as required. ∎

7. Spectrum of a Multiplication Operator on $L^2(\mathbb{R},\mathbb{C}^2)$. The essential spectrum of the asymptotic linearized operator, $\mathcal{M}_\infty$, is equal to the spectrum of its Fourier transform, $\widehat{\mathcal{M}}_\infty$, which is a multiplication operator on $L^2(\mathbb{R},\mathbb{C}^2)$. In this section, we derive a formula for the spectrum of a general class of multiplication operators on $L^2(\mathbb{R},\mathbb{C}^2)$. The proof is based on that of a similar well-known formula for multiplication operators on $L^2(\mathbb{R},\mathbb{C})$ [5, Prop. 4.2].

Definition 7.1. *Let $\boldsymbol{Q}:\mathbb{R}\to\mathbb{C}^{2\times2}$. The* multiplication operator, *$\mathcal{M}_{\boldsymbol{Q}}$, induced on $L^2(\mathbb{R},\mathbb{C}^2)$ by $\boldsymbol{Q}$ is defined by*

$$
(\mathcal{M}_{\boldsymbol{Q}}\boldsymbol{w})(x):=\boldsymbol{Q}(x)\boldsymbol{w}(x)\ \text{for all } \boldsymbol{w} \text{ in the domain} \tag{7.1}
$$

$$
D(\mathcal{M}_{\boldsymbol{Q}})=\{\boldsymbol{w}\in L^2(\mathbb{R},\mathbb{C}^2)\,:\,\boldsymbol{Q}\boldsymbol{w}\in L^2(\mathbb{R},\mathbb{C}^2)\}. \tag{7.2}
$$

Proposition 7.2. *If $\boldsymbol{Q}\in L^\infty(\mathbb{R},\mathbb{C}^{2\times2})$, then $\mathcal{M}_{\boldsymbol{Q}}$ is everywhere defined, bounded and closed, with*

$$
\|\mathcal{M}_{\boldsymbol{Q}}\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2)))}\le\|\boldsymbol{Q}\|_\infty\,, \tag{7.3}
$$

where

$$\|\boldsymbol{Q}\|_\infty := \sup_{x\in\mathbb{R}} \|\boldsymbol{Q}(x)\|_{\mathbb{C}^{2\times 2}}\,. \tag{7.4}$$

We now state the main result of this section.

THEOREM 7.3. *Let $\boldsymbol{Q} \in L^\infty(\mathbb{R},\mathbb{C}^{2\times 2}) \cap C^0(\mathbb{R},\mathbb{C}^{2\times 2})$. If $\|\boldsymbol{Q}(x)\|_{\mathbb{C}^{2\times 2}} \to 0$ as $x \to \pm\infty$, then the spectrum of $\mathcal{M}_{\boldsymbol{Q}}$ is given by*

$$\begin{aligned}\sigma(\mathcal{M}_{\boldsymbol{Q}}) &= \{\lambda \in \mathbb{C} : \exists x \in \mathbb{R} \text{ such that } \det(\lambda \boldsymbol{I} - \boldsymbol{Q}(x)) = 0\} \cup \{0\}\\ &= \{\lambda \in \mathbb{C} : \exists x \in \mathbb{R} \text{ such that } \lambda \in \sigma(\boldsymbol{Q}(x))\} \cup \{0\}.\end{aligned} \tag{7.5}$$

The proof of Theorem 7.3 relies on several preliminary results. First, Proposition 7.2 can be improved upon as follows.

PROPOSITION 7.4. *Suppose that $\boldsymbol{Q} \in C^0(\mathbb{R},\mathbb{C}^{2\times 2})$. Then, the operator $\mathcal{M}_{\boldsymbol{Q}}$ is bounded if and only if $\boldsymbol{Q}$ is bounded. In this case,*

$$\|\mathcal{M}_{\boldsymbol{Q}}\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2)))} = \|\boldsymbol{Q}\|_\infty\,. \tag{7.6}$$

The proof of this proposition relies on the following well-known result on the Dirac delta distribution.

LEMMA 7.5. *Let $g \in L^1(\mathbb{R})$ with $\int_{\mathbb{R}} g(x)dx = 1$. Set $g_{s,\delta}(x) = \frac{1}{\delta} g\left(\frac{x-s}{\delta}\right)$, where $\delta > 0$. Then $\lim_{\delta\to 0} \int_{\mathbb{R}} \phi(x) g_{s,\delta}(x)dx = \phi(s)$ for all $\phi \in L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$. That is, for every $\epsilon > 0$, there exists $\widetilde{\delta} = \widetilde{\delta}(\epsilon,\phi)$ such that*

$$\phi(s) - \epsilon \le \int_{\mathbb{R}} \phi(x) g_{s,\delta}(x)dx \le \phi(s) + \epsilon, \quad \text{whenever } \delta \le \widetilde{\delta}. \tag{7.7}$$

Proof of Proposition 7.4. If $\mathbf{Q}$ is bounded, then $\mathcal{M}_{\mathbf{Q}}$ is bounded by Proposition 7.2. Conversely, suppose $\mathcal{M}_{\mathbf{Q}}$ is bounded. Then,

$$\|\mathcal{M}_{\mathbf{Q}}\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \ge \|\mathcal{M}_{\mathbf{Q}} \boldsymbol{w}\|_{L^2(\mathbb{R},\mathbb{C}^2)}\,, \tag{7.8}$$

for all $\boldsymbol{w} \in L^2(\mathbb{R},\mathbb{C}^2)$ with $\|\boldsymbol{w}\|_{L^2(\mathbb{R},\mathbb{C}^2)} = 1$. Fix $s \in \mathbb{R}$ and choose $\boldsymbol{w}(x) = \boldsymbol{w}_{s,\delta}(x) = \sqrt{g_{s,\delta}(x)}\boldsymbol{v}(x)$, for some vector $\boldsymbol{v}(x) \in \mathbb{C}^2$ and where $g_{s,\delta}$ is as in Proposition 7.5. If we require that $\|\boldsymbol{v}(x)\|_{\mathbb{C}^2} = 1$ for all x, then $\|\boldsymbol{w}\|_{L^2(\mathbb{R},\mathbb{C}^2)} = 1$ holds. Furthermore, for each x, we can chose $\boldsymbol{v}(x)$ so that

$$\|\mathbf{Q}(x)\boldsymbol{v}(x)\|_{\mathbb{C}^2} = \|\mathbf{Q}(x)\|_{\mathbb{C}^{2\times 2}}\,. \tag{7.9}$$

Then

$$\|\mathcal{M}_{\mathbf{Q}}\|^2_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \ge \|\mathcal{M}_{\mathbf{Q}} \boldsymbol{w}_{s,\delta}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} = \int_{\mathbb{R}} \|\mathbf{Q}(x)\|^2_{\mathbb{C}^{2\times 2}}\; g_{s,\delta}(x)dx.$$

Let $\epsilon > 0$. Choosing $\phi(x) = \|\mathbf{Q}(x)\|^2_{\mathbb{C}^{2\times 2}}$ in Proposition 7.5 we find that there exists $\widetilde{\delta} = \widetilde{\delta}(\epsilon, s) > 0$ so that for all $\delta < \widetilde{\delta}$

$$\|\mathcal{M}_{\mathbf{Q}}\|^2_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \ge \int_{\mathbb{R}} \|\mathbf{Q}(x)\|^2_{\mathbb{C}^{2\times 2}}\, g_{s,\delta}(x)dx > \|\mathbf{Q}(s)\|^2_{\mathbb{C}^{2\times 2}} - \epsilon. \tag{7.10}$$

Therefore,

$$\|\mathbf{Q}\|_\infty = \sup_{s\in\mathbb{R}} \|\mathbf{Q}(s)\|_{\mathbb{C}^{2\times 2}} \le \|\mathcal{M}_{\mathbf{Q}}\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))}\,, \tag{7.11}$$

and so $\mathbf{Q}$ is bounded, and (7.6) holds by Proposition 7.2. ∎

Next, in Proposition 7.6 and Proposition 7.7 we state some properties of a matrix valued function, $\mathbf{Q} \in L^\infty(\mathbb{R}, \mathbb{C}^{2\times 2})$, which are used in the proof of Proposition 7.10 below.

PROPOSITION 7.6. *Let $\boldsymbol{Q} : \mathbb{R} \to \mathbb{C}^{2\times 2}$ be continuous with $\|\boldsymbol{Q}\|_\infty < \infty$ and suppose that $0 \notin \overline{\mathrm{Im}(\det \boldsymbol{Q})}$. Then $\boldsymbol{Q}^{-1} : \mathbb{R} \to \mathbb{C}^{2\times 2}$ is continuous and $\left\|\boldsymbol{Q}^{-1}\right\|_\infty < \infty$.*

Proof. Since, $0 \notin \overline{\mathrm{Im}(\det \mathbf{Q})}$, there exists $\epsilon > 0$ such that $|\det \mathbf{Q}(x)| > \epsilon$, for all $x \in \mathbb{R}$. So,

$$\mathbf{Q}^{-1}(x) = \frac{1}{\det Q(x)} \begin{bmatrix} Q_{22}(x) & -Q_{12}(x) \\ -Q_{21}(x) & Q_{11}(x) \end{bmatrix}$$

exists and is continuous. Furthermore,

$$\left\|\mathbf{Q}^{-1}(x)\right\|_{\mathbb{C}^{2\times 2}}^2 \le \left\|\mathbf{Q}^{-1}(x)\right\|_F^2 = \frac{\|\mathbf{Q}(x)\|_F^2}{|\det \mathbf{Q}(x)|^2} \le \frac{4\,\|\mathbf{Q}(x)\|_{\mathbb{C}^{2\times 2}}^2}{|\det \mathbf{Q}(x)|^2} \le \frac{4}{\epsilon^2}\,\|\mathbf{Q}\|_\infty^2 . \qquad \square \tag{7.12}$$

PROPOSITION 7.7. *Let $\boldsymbol{Q} \in \mathbb{C}^{2\times 2}$ be a matrix. Then there exists a vector $\boldsymbol{u} \in \mathbb{C}^2$ with $\|\boldsymbol{u}\|_{\mathbb{C}^2} = 1$ so that*

$$\|\boldsymbol{Q}\boldsymbol{u}\|_{\mathbb{C}^2}^2 \le |\det \boldsymbol{Q}|. \tag{7.13}$$

REMARK. *Geometrically $\boldsymbol{Q}$ changes areas by a factor of $|\det\ \boldsymbol{Q}|$. This result says there exists a direction $\boldsymbol{u}$ in which $\boldsymbol{Q}$ changes lengths by at most $\sqrt{|\det\ \boldsymbol{Q}|}$.*

Proof. The following self evident claims leads to the proof of (7.13).

CLAIM 7.8. *Let $\boldsymbol{Q} = \boldsymbol{U}\boldsymbol{R}$ be the QR decomposition of $\boldsymbol{Q}$, where $\boldsymbol{U}$ is unitary and $\boldsymbol{R}$ is upper triangular. Suppose (7.13) holds for $\boldsymbol{R}$, then it also holds for $\boldsymbol{Q}$.*

CLAIM 7.9. *Suppose $\boldsymbol{Q} = \alpha\widetilde{\boldsymbol{Q}}$ for some $\alpha \in \mathbb{C}$ and that the (7.13) holds for $\widetilde{\boldsymbol{Q}}$. Then (7.13) also holds for $\boldsymbol{Q}$.*

By Claim 7.8 it suffices to establish (7.13) for $\mathbf{R} = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$.

Case I: If $a = 0$, let $\boldsymbol{u} = (1, 0)$. Then $\mathbf{R}\boldsymbol{u} = (0, 0)$. Hence, $\|\mathbf{R}\boldsymbol{u}\|_{\mathbb{C}^2}^2 = 0 = |\det \mathbf{R}|$, and so (7.13) holds.

Case II: If $a \neq 0$, then by Claim 7.9 we just need to show that (7.13) holds for matrices $\widetilde{\mathbf{R}}$ of the form $\widetilde{\mathbf{R}} = \begin{bmatrix} 1 & b \\ 0 & d \end{bmatrix}$. If $|d| \ge 1$, we choose $\boldsymbol{u} = (1, 0)$ to obtain $\left\|\widetilde{\mathbf{R}}\boldsymbol{u}\right\|_{\mathbb{C}^2}^2 = 1 \le |d| = |\det \widetilde{\mathbf{R}}|$. Finally, if $|d| < 1$, choosing $\boldsymbol{u} = \left(-b/\sqrt{1 + |b|^2}, 1/\sqrt{1 + |b|^2}\right)$ we obtain $\widetilde{\mathbf{R}}\boldsymbol{u} = (0, d)/\sqrt{1 + |b|^2}$. Hence, $\left\|\widetilde{\mathbf{R}}\boldsymbol{u}\right\|_{\mathbb{C}^2}^2 = |d|^2/(1 + |b|^2) \le |d|^2 \le |d| = |\det \widetilde{\mathbf{R}}|$. $\square$

PROPOSITION 7.10. *Let $\boldsymbol{Q} : \mathbb{R} \to \mathbb{C}^{2\times 2}$ be continuous with $\|\boldsymbol{Q}\|_\infty < \infty$. Then the operator $\mathcal{M}_{\boldsymbol{Q}}$ has a bounded inverse if and only if $0 \notin \overline{Im(det\boldsymbol{Q})}$. In that case, $\boldsymbol{Q}$ has a bounded inverse, $\boldsymbol{Q}^{-1}$, and*

$$\mathcal{M}_{\boldsymbol{Q}}^{-1} = M_{\boldsymbol{Q}^{-1}}.$$

Proof. Suppose $0 \notin \overline{\mathrm{Im}(\det \mathbf{Q})}$. By Proposition 7.6, $\left\|\mathbf{Q}^{-1}\right\|_\infty \leq \infty$. Hence, by Proposition 7.4, $\mathcal{M}_{\mathbf{Q}}^{-1}$ is bounded and

$$\left\|\mathcal{M}_{\mathbf{Q}}^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} = \left\|\mathbf{Q}^{-1}\right\|_\infty \leq \infty. \tag{7.14}$$

Conversely, suppose that $\mathcal{M}_{\mathbf{Q}}$ has a bounded inverse. Then for all $\boldsymbol{w} \in L^2(\mathbb{R},\mathbb{C}^2)$,

$$\gamma := \frac{1}{\left\|\mathcal{M}_{\mathbf{Q}}^{-1}\right\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))}} \leq \frac{\|\mathcal{M}_{\mathbf{Q}}\boldsymbol{w}\|_{L^2(\mathbb{R},\mathbb{C}^2)}}{\|\boldsymbol{w}\|_{L^2(\mathbb{R},\mathbb{C}^2)}}. \tag{7.15}$$

We will show that for all $x \in \mathbb{R}$

$$|\det \mathbf{Q}(x)| > \frac{\gamma^2}{8}, \tag{7.16}$$

and hence $0 \notin \overline{\mathrm{Im}(\det \mathbf{Q})}$.

Assume for the sake of contradiction that there exists $s \in \mathbb{R}$ such that

$$|\det \mathbf{Q}(s)| \leq \frac{\gamma^2}{8}. \tag{7.17}$$

Let $\boldsymbol{w}(x) = \boldsymbol{w}_{s,\delta}(x) = \sqrt{g_{s,\delta}(x)}\boldsymbol{u}(x)$, where $g_{s,\delta}(x)$ is as in Proposition 7.5 and, using Proposition 7.7, for each $x \in \mathbb{R}$, $\boldsymbol{u}(x) \in \mathbb{C}^2$ is chosen so that $\|\boldsymbol{u}(x)\|_{\mathbb{C}^2} = 1$ and

$$\|\mathbf{Q}(x)\boldsymbol{u}(x)\|_{\mathbb{C}^2}^2 \leq |\det \mathbf{Q}(x)|. \tag{7.18}$$

Let $\epsilon > 0$. By (7.18) and Proposition 7.5 there exists $\delta > 0$ so that

$$\begin{aligned}\|\mathcal{M}_{\mathbf{Q}}\boldsymbol{w}_{s,\delta}\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 &= \int_{\mathbb{R}} \left\|\mathbf{Q}(x)\sqrt{g_{s,\delta}(x)}\boldsymbol{u}(x)\right\|_{\mathbb{C}^2}^2 dx \leq \int_{\mathbb{R}} g_{s,\delta}(x)\,|\det \mathbf{Q}(x)|\,dx \\ &< |\det \mathbf{Q}(x)| + \epsilon < \frac{\gamma^2}{8} + \epsilon.\end{aligned}$$

Choosing $\epsilon = \frac{\gamma^2}{8}$ and applying our assumption (7.17) we find that

$$\|\mathcal{M}_{\mathbf{Q}}\boldsymbol{w}_{s,\delta}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \leq \frac{\gamma}{2}, \tag{7.19}$$

which is a contradiction to (7.15). Therefore, for all $x \in \mathbb{R}$ $|\det \mathbf{Q}(x)| > \frac{\gamma^2}{8}$. Hence, $0 \notin \overline{\mathrm{Im}(\det \mathbf{Q})}$. Finally, using (7.6), we conclude that $\left\|\mathbf{Q}^{-1}\right\|_\infty \leq \infty$. ∎

Proof of Theorem 7.3. By Proposition 7.10

$$\begin{aligned}\lambda \in \rho(\mathcal{M}_{\mathbf{Q}}) &\iff M_{\lambda-Q} \text{ has a bounded inverse} \\ &\iff 0 \notin \overline{\mathrm{Im}(\det(\lambda\mathbf{I} - \mathbf{Q}))} \\ &\iff \exists\epsilon > 0 \text{ such that } \forall x \in \mathbb{R} \quad |\det(\lambda\mathbf{I} - \mathbf{Q}(x))| \geq \epsilon.\end{aligned}$$

Therefore,

$$\begin{aligned}\lambda \in \sigma(\mathcal{M}_{\mathbf{Q}}) &\iff \lambda \notin \rho(\mathcal{M}_{\mathbf{Q}}) \\ &\iff \forall\epsilon > 0\, \exists x \in \mathbb{R} \text{ such that } |\det(\lambda\mathbf{I} - \mathbf{Q}(x))| < \epsilon.\end{aligned} \tag{7.20}$$

Let

$$\widetilde{\sigma}(\mathcal{M}_{\mathbf{Q}}) = \{\lambda \in \mathbb{C} : \exists x \in \mathbb{R} \text{ such that } \det(\lambda\mathbf{I} - \mathbf{Q}(x)) = 0\}. \tag{7.21}$$

Then $\widetilde{\sigma}(\mathcal{M}_{\mathbf{Q}}) \subseteq \sigma(\mathcal{M}_{\mathbf{Q}})$. Let $\lambda \in \sigma(\mathcal{M}_{\mathbf{Q}})\backslash\widetilde{\sigma}(\mathcal{M}_{\mathbf{Q}})$. To complete the proof, we must show $\lambda = 0$. Choosing $\epsilon = 1/n$ in (7.20),

$$\exists x_n \in \mathbb{R} \text{ such that } \det(\lambda\mathbf{I} - \mathbf{Q}(x_n)) \leq 1/n. \tag{7.22}$$

Suppose that the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded. Then there exists a convergent subsequence $x_{n_k} \to x_*$. Since, we are assuming that $\mathbf{Q}$ is continuous,

$$\det(\lambda\mathbf{I} - \mathbf{Q}(x_*)) = \lim_{n\to\infty} \det(\lambda\mathbf{I} - \mathbf{Q}(x_n)) = 0. \tag{7.23}$$

Therefore, $\lambda \in \widetilde{\sigma}(\mathcal{M}_{\mathbf{Q}})$, which is a contradiction. Hence, x_n is not bounded and so

$$\exists x_n \to \infty \text{ such that } \|\mathbf{Q}(x_n)\|_{\mathbb{C}^{2\times 2}} \to 0. \tag{7.24}$$

Let $a_n = \det(\lambda\mathbf{I} - \mathbf{Q}(x_n)) = \lambda^2 - \operatorname{trace}(\mathbf{Q}(x_n))\lambda + \det(\mathbf{Q}(x_n))$. Therefore,

$$\lambda = \frac{1}{2}\left[\operatorname{trace}(\mathbf{Q}(x_n)) \pm \sqrt{\operatorname{trace}^2(\mathbf{Q}(x_n)) - 4(\det(\mathbf{Q}(x_n)) - a_n)}\right]. \tag{7.25}$$

Now, by (7.22), $a_n \to 0$ and by assumption $\|\mathbf{Q}(x_n)\|_F \to 0$ as $n \to \infty$. Therefore, $\lambda = 0$ must hold. ∎

8. The Essential Spectrum of the Asymptotic Monodromy Operator. In this section we prove Theorem 4.6 which gives the formula for the essential spectrum of $\mathcal{M}_\infty$. The proof relies on the following two results.

LEMMA 8.1. *Let* $\mathbf{A}(a,b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. *Then*

$$e^{\mathbf{A}(a,b)} = e^a \boldsymbol{R}(b), \tag{8.1}$$

where $\boldsymbol{R}(b) = \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$ *is a rotation matrix.*

Proof. Diagonalize $\mathbf{A}(a,b)$ and use Euler's formula. ∎

Next, working with Definition 4.5, we have the following result.

PROPOSITION 8.2. *Let* $\mathcal{M}_\infty : L^2(\mathbb{R},\mathbb{C}^2) \to L^2(\mathbb{R},\mathbb{C}^2)$ *be the asymptotic monodromy operator given by* (3.9). *Then*

$$\sigma_{\text{ess}}(\mathcal{M}_\infty) = \sigma_{\text{ess}}(\widehat{\mathcal{M}}_\infty), \tag{8.2}$$

where

$$\widehat{\mathcal{M}}_\infty = \mathcal{F} \circ \mathcal{M}_\infty \circ \mathcal{F}^{-1}. \tag{8.3}$$

Here, $\mathcal{F} : L^2(\mathbb{R},\mathbb{C}^2) \to L^2(\mathbb{R},\mathbb{C}^2)$ *is the Fourier transform.*

Proof of Theorem 4.6. By Proposition 8.2 it suffices to compute $\sigma_{\text{ess}}(\widehat{\mathcal{M}}_\infty)$. First, we show that

$$\widehat{\mathcal{M}}_\infty = \widehat{\mathcal{U}}_\infty^{\text{OC}} \circ \widehat{\mathcal{U}}_\infty^{\text{DCF}} \circ \widehat{\mathcal{U}}_\infty^{\text{SMF2}} \circ \widehat{\mathcal{U}}_\infty^{\text{FA}} \circ \widehat{\mathcal{U}}_\infty^{\text{SMF1}} \circ \widehat{\mathcal{U}}_\infty^{\text{SA}} \tag{8.4}$$

is a multiplication operator by showing that each transfer function $\widehat{\mathcal{U}}_\infty$ is a multiplication operator. Here, for each laser component the transfer function $\widehat{\mathcal{U}}_\infty$ is the Fourier transform of the asymptotic linearized transfer function, $\mathcal{U}_\infty$, given in Section 3. We then use Theorem 7.3 to obtain $\sigma_{\rm ess}(\widehat{\mathcal{M}}_\infty)$.

For the saturable absorber,

$$(\widehat{\mathcal{U}}^{\rm SA}_\infty \widehat{\boldsymbol{u}}_{\rm in})(\omega) = (1-\ell_0)\widehat{\boldsymbol{u}}_{\rm in}(\omega), \tag{8.5}$$

and, as in the derivation of (2.9), for the dispersion compensation element,

$$(\widehat{\mathcal{U}}^{\rm DCF}_\infty \widehat{\boldsymbol{u}}_{\rm in})(\omega) = \exp\left\{\mathbf{A}\left(0, \frac{\omega^2}{2}\beta_{\rm DCF}\right)\right\}\widehat{\boldsymbol{u}}_{\rm in}(\omega). \tag{8.6}$$

For the two single mode fiber segments, a similar formula holds for each solution operator, $\widehat{\mathcal{U}}^{\rm SMF}_\infty$, but with $\beta_{\rm DCF}$ replaced by $\beta_{\rm SMF}{\rm L}_{\rm SMF}$. For the fiber amplifier,

$$(\widehat{\mathcal{U}}^{\rm FA}_\infty \widehat{\boldsymbol{u}}_{\rm in})(\omega) = \exp\left\{\mathbf{A}\left(\frac{1}{2}\left(1-\frac{\omega^2}{\Omega_g^2}\right)\int_0^{L_{\rm FA}} g(t)dt,\ \frac{\omega^2}{2}\beta_{\rm FA}{\rm L}_{\rm FA}\right)\right\}\widehat{\boldsymbol{u}}_{\rm in}(\omega). \tag{8.7}$$

Finally, $\widehat{\mathcal{U}}^{\rm OC}_\infty = \mathcal{P}^{\rm OC}$, which is given by (2.11).

Combining these formulae, applying Lemma 8.1, and using the fact that $\mathbf{R}(\theta_1)\circ\mathbf{R}(\theta_2) = \mathbf{R}(\theta_1+\theta_2)$ we have

$$(\widehat{\mathcal{M}}_\infty \widehat{\boldsymbol{u}}_{\rm in})(\omega) = \widehat{\mathbf{M}}_\infty(\omega)\widehat{\boldsymbol{u}}_{\rm in}(\omega), \tag{8.8}$$

where

$$\widehat{\mathbf{M}}_\infty(\omega) = \frac{(1-\ell_0)}{\sqrt{2}}\exp\left\{\frac{1}{2}\left(1-\frac{\omega^2}{\Omega_g^2}\right)\int_0^{L_{\rm FA}} g(t)dt\right\}\mathbf{R}\left(\frac{\omega^2}{2}\beta_{\rm RT}\right). \tag{8.9}$$

Using Theorem 7.3 with $\mathbf{Q} = \widehat{\mathbf{M}}_\infty(\omega)$, we obtain

$$\begin{gathered}\sigma(\mathcal{M}_\infty) = \{\,\lambda_\pm(\omega)\in\mathbb{C}\mid \omega\in\mathbb{R}\,\}\cup\{0\},\\ \lambda_\pm(\omega) = \frac{(1-\ell_0)}{\sqrt{2}}\exp\left\{\frac{1}{2}\left(1-\frac{\omega^2}{\Omega_g^2}\right)\int_0^{L_{\rm FA}} g(t)dt\right\}\exp\left\{\pm i\frac{\omega^2}{2}\beta_{\rm RT}\right\}.\end{gathered} \tag{8.10}$$

Finally we show that $\sigma_{\rm pt}(\mathcal{M}_\infty) = \phi$, from which it follows that $\sigma_{\rm ess}(\mathcal{M}_\infty) = \sigma(\mathcal{M}_\infty)$. For this we recall that the point spectrum of a multiplication operator such as $\widehat{\mathcal{M}}_\infty$ is given by [5]

$$\sigma_{\rm pt}(\widehat{\mathcal{M}}_\infty) = \left\{\lambda\in\mathbb{C}\,:\,\mu\left\{\omega\in\mathbb{R}\,:\,\det[\widehat{\mathbf{M}}_\infty(\omega)-\lambda]=0\right\}>0\right\}, \tag{8.11}$$

where μ denotes Lebesgue measure on $\mathbb{R}$. Therefore, to show that $\sigma_{\rm pt}(\widehat{\mathcal{M}}_\infty) = \phi$, we must show for all $\lambda\in\mathbb{C}$ that the set

$$S_\lambda = \{\omega\in\mathbb{R}\,:\,\lambda_+(\omega)=\lambda \text{ or } \lambda_-(\omega)=\lambda\}, \tag{8.12}$$

has measure zero. We observe that $\lambda_\pm:\mathbb{R}\to\mathbb{C}$ generically parametrizes a pair of counter-rotating spirals. Invoking the assumptions of the theorem, since $\ell_0\neq 1$, and either $\beta_{\rm RT}\neq 0$ or $\Omega_g<\infty$ and $\int_0^{L_{\rm FA}} g(t)dt\neq 0$, the mappings $\lambda_\pm:\mathbb{R}\to\mathbb{C}$ are at most countable-to-one, which implies that S_λ has measure zero for all $\lambda\in\mathbb{C}$. ∎

9. Relative compactness for the linearized differential operators in the fiber amplifier. In this section we show that the linearized differential operator in the fiber amplifier, $\mathcal{L}(t)$, is a relatively compact perturbation of the asymptotic linearized differential operator, $\mathcal{L}_\infty(t)$, provided that the nonlinear pulse satisfies some reasonable weak regularity and exponential decay assumptions.

By (3.3), the operators $\mathcal{L}(t)$ and $\mathcal{L}_\infty(t)$ are related by

$$\mathcal{L}(t) \;=\; \mathcal{L}_\infty(t) \;+\; \mathbf{M}(t), \tag{9.1}$$

where

$$\mathcal{L}_\infty(t) \;=\; \mathbf{B}\left(\frac{g(t)}{2\Omega_g^2}, \frac{\beta}{2}\right)\partial_x^2 \;+\; \frac{1}{2}g(t)\mathcal{I}, \tag{9.2}$$

with $\mathbf{B}(a,b) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, and where $\mathbf{M}(t)$ is the matrix-valued multiplication operator

$$\mathbf{M}(t,\cdot)\boldsymbol{u} = \mathbf{M}_1(t,\cdot)\boldsymbol{u} - \boldsymbol{\phi}(t,\cdot)\langle\boldsymbol{\psi}(t,\cdot),\boldsymbol{u}\rangle, \tag{9.3}$$

$$\mathbf{M}_1(t,\cdot) = \gamma|\boldsymbol{\psi}(t,\cdot)|^2\mathbf{J} + 2\gamma\mathbf{J}\boldsymbol{\psi}(t,\cdot)\boldsymbol{\psi}^T(t,\cdot). \tag{9.4}$$

Here $\boldsymbol{\psi}$ is the pulse about which the Haus master equation (2.5) is linearized and $\boldsymbol{\phi}$ is given by (6.6). Note that here we have chosen $\mathbf{M}$ so that $\mathbf{M}(t,x) \to \mathbf{0}$ as $x \to \pm\infty$.

THEOREM 9.1. *Assume that Hypothesis 4.3 is met and that $(g_0/\Omega_g, \beta) \neq (0,0)$. Then, the differential operator, $\mathcal{L}(t)$, given in (9.1), is a relatively compact perturbation of $\mathcal{L}_\infty(t)$ in that there exists a $\lambda \in \rho(\mathcal{L}_\infty)$ so that the operator $\mathbf{M} \circ (\mathcal{L}_\infty - \lambda)^{-1}$ on $L^2(\mathbb{R}, \mathbb{C}^2)$ is compact.*

Proof. Using an idea of Kapitula, Kutz, and Sandstede [16] in their paper on the Evans function for nonlocal equations, we observe that

$$\mathcal{L} = \mathcal{L}_\infty + \mathbf{M}_1 + \mathcal{K} \circ \mathcal{J}, \tag{9.5}$$

where $\mathcal{J} : L^2(\mathbb{R}, \mathbb{C}^2) \to \mathbb{C}$ is given by $\mathcal{J}(\boldsymbol{u}) \;=\; \langle\boldsymbol{\psi}(t,\cdot), \boldsymbol{u}\rangle$, and $\mathcal{K} : \mathbb{C} \to L^2(\mathbb{R}, \mathbb{C}^2)$ is given by $\mathcal{K}(a) \;=\; a\boldsymbol{\phi}$. Under Hypothesis 4.3, the analogous result in Zweck et al. [39, Theorem 3.1] guarantees that $\mathcal{L}_\infty + \mathbf{M}_1$ is a relatively compact perturbation of $\mathcal{L}_\infty$. The theorem now follows from the fact that $\mathcal{K} \circ \mathcal{J}$ is compact, since it factors through the finite dimensional space, $\mathbb{C}$. ◻

10. Analyticity of asymptotic linearized operator in the fiber amplifier. In this section, we show that the operator $\mathcal{L}_\infty(t)\mathcal{U}_\infty(t,s)$ is bounded on $L^2(\mathbb{R}, \mathbb{C}^2)$, where $\mathcal{L}_\infty(t)$ is the asymptotic linearized operator in the fiber amplifier given by (9.2), and $\mathcal{U}_\infty(t,s)$ is the corresonding evolution family. Zweck et al. [39] previously established an analogous result for the constant-coefficient complex Ginzburg-Landau equation under the assumption that the spectral filtering coefficient in the equation is positive. These results will be used in Section 11 to prove our main result, Theorem 4.7.

We begin by recalling what it means for an operator to be sectorial [24, 27].

DEFINITION 10.1. *A linear operator $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is* sectorial *if $\exists\ \omega \in \mathbb{R}$, $\theta \in (\pi/2, \pi]$, $M > 0$ so that*

1. $\rho(\mathcal{A}) \supset S_{\theta,\omega} := \{\lambda \in \mathbb{C} \,|\, \lambda \neq \omega,\ |\arg(\lambda - \omega)| < \theta\}$, *and*
2. $\|\mathcal{R}(\lambda : \mathcal{A})\| \leq \frac{M}{|\lambda - \omega|}$, *for all $\lambda \in S_{\theta,\omega}$.*

REMARK. *Lunardi [24, Chapter 2] shows that if $\mathcal{A}$ is a sectorial operator then a family of operators, $\mathcal{T}(t) = e^{t\mathcal{A}}$, for $t > 0$, can be defined in terms of a Dunford contour integral so as to satisfy the semigroup properties*

1. $\mathcal{T}(0) = \mathcal{I}$,
2. $\mathcal{T}(s+t) = \mathcal{T}(s)\mathcal{T}(t)$, *for all* $T, s \geq 0$,

and for which the mapping $t \mapsto e^{t\mathcal{A}} : \mathbb{R}^+ \to \mathcal{B}(X)$ *is analytic. Furthermore,*

$$\frac{d}{dt} e^{t\mathcal{A}} = \mathcal{A}\, e^{t\mathcal{A}}. \tag{10.1}$$

Such a semigroup is called an analytic semigroup.

We consider solutions, $\boldsymbol{u} : [s, L_{\mathrm{FA}}] \to H^2(\mathbb{R}, \mathbb{C}^2)$, of the initial value problem

$$\begin{aligned} \partial_t \boldsymbol{u} &= \mathcal{L}_\infty(t)\boldsymbol{u}, \text{ for } t > s, \\ \boldsymbol{u}(s) &= \boldsymbol{v}, \qquad \text{for } \boldsymbol{v} \in H^2(\mathbb{R}, \mathbb{C}^2). \end{aligned} \tag{10.2}$$

THEOREM 10.2. *Suppose that* $0 < \Omega_g < \infty$, *that* $(g_0, \beta) \neq (0,0)$, *and that* $\boldsymbol{\psi}$ *is differentiable with respect to* t. *Then, there exists a unique evolution system,* $\mathcal{U}_\infty(t,s)$, *for* (10.2) *with* $0 \leq s \leq t \leq L_{\mathrm{FA}}$ *so that*

1. $\exists\, C$ *so that for all* s, t *we have* $\|\mathcal{U}_\infty(t,s)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))} \leq C$,
2. $\mathcal{U}_\infty(s,s) = \mathcal{I}$ *and* $\mathcal{U}_\infty(t,r) = \mathcal{U}_\infty(t,s) \circ \mathcal{U}_\infty(s,r)$ *for all* $0 \leq r \leq s < t \leq L_{\mathrm{FA}}$,
3. $\mathcal{U}_\infty(t,s) \in \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2), H^2(\mathbb{R},\mathbb{C}^2))$,
4. *The mapping* $t \mapsto \mathcal{U}_\infty(t,s)$ *is differentiable for* $t \in (s, L_{FA}]$ *with values in* $\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))$, *and* $\partial_t \mathcal{U}_\infty(t,s) = \mathcal{L}_\infty(t)\mathcal{U}_\infty(t,s)$, *i.e., the function* $\boldsymbol{u}(t) = \mathcal{U}_\infty(t,s)\boldsymbol{v}$ *solves* (10.2), *and*
5. $\exists\, C_1$ *and* C_2 *so that* $\forall\, 0 \leq s < t \leq L_{\mathrm{FA}}$,

$$\|\mathcal{L}_\infty(t)\mathcal{U}_\infty(t,s)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2)} \;\leq\; C_1 \frac{G(t,s)}{t-s} + C_2 \frac{g(t)}{2}, \tag{10.3}$$

where $G(t,s) = \exp\left(\frac{1}{2}\int_s^t g(\tau)\, d\tau\right)$.

Proof. We will show that the first four conclusions of the theorem hold for the evolution operator, $\mathcal{V}_\infty(t,s)$, associated to the differential operator, $\mathbf{B}(t)\partial_x^2$, and that

$$\|(\mathbf{B}(t)\partial_x^2)\mathcal{V}_\infty(t,s)\|_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2)} \;\leq\; \frac{C_1}{t-s}. \tag{10.4}$$

Then, the theorem immediately follows for the original operators $\mathcal{L}_\infty(t) = \mathbf{B}(t)\partial_x^2 + \frac{1}{2}g(t)\mathcal{I}$ with $\mathcal{U}_\infty(t,s) = G(t,s)\mathcal{V}_\infty(t,s)$. Applying a result from Lunardi [24, Chap. 6], to establish the result for $\mathcal{V}_\infty(t,s)$ it suffices to show that the operator $\mathcal{A} = \mathcal{A}(t) := \mathbf{B}(t)\partial_x^2$ is sectorial and that $t \mapsto \mathcal{A}(t) \in \mathrm{Lip}([0, L_{\mathrm{FA}}], \mathcal{B}(H^2(\mathbb{R},\mathbb{C}^2), L^2(\mathbb{R},\mathbb{C}^2)))$.

To show $\mathcal{A}$ is sectorial, we first observe that $\mathcal{A}$ is closed and that $\exists\, \omega \geq 0$ so that $\forall \lambda > \omega$, $\lambda \in \rho(\mathcal{A})$ and $\|\mathcal{R}(\lambda : \mathcal{A})\| \leq \frac{1}{\lambda - \omega}$. Therefore, by [27, Cor 1.3.8], $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup for which $\|\mathcal{T}(t)\| \leq e^{\omega t}$. By the proof of [39, Lemma 5.2], for all $\sigma > 0$,

$$\|\mathcal{R}(\sigma + i\tau : \mathcal{A})\| \leq \frac{C}{|\tau|}. \tag{10.5}$$

To show that this condition implies that $\mathcal{A}$ is sectorial we make use of [27, Thm 2.5.2]. However, as stated, this theorem requires that the semigroup $\mathcal{T}(t)$ is uniformly bounded and $0 \in \rho(\mathcal{A})$. Since neither of these conditions is guaranteed to hold, we proceed as follows. Fix $\epsilon > 0$, define $\mathcal{T}_\epsilon(t) := e^{-(\epsilon+\omega)t}\mathcal{T}(t)$, and let $\mathcal{A}_\epsilon = \frac{\partial \mathcal{T}_\epsilon}{\partial t}(0)$. Then $\|\mathcal{T}_\epsilon(t)\| < 1$ is uniformly bounded and $0 \in \rho(\mathcal{A}_\epsilon)$. Therefore, the assumptions of [27, Thm 2.5.2] hold for $\mathcal{A}_\epsilon$. Furthermore, (10.5) holds for $\mathcal{A}_\epsilon$ since $\mathcal{R}(\sigma + i\tau : \mathcal{A}_\epsilon) = \mathcal{R}(\sigma + \epsilon + \omega + i\tau : \mathcal{A})$. So by [27, Thm 2.5.2], $\exists\ 0 < \delta < \frac{\pi}{2}$, $M > 0$ so that

1. $\rho(\mathcal{A}_\epsilon) \supset \Sigma = \{\lambda \in \mathbb{C} : |\arg\lambda| < \frac{\pi}{2} + \delta\} \cup \{0\}$, and
2. $\|\mathcal{R}(\lambda : \mathcal{A}_\epsilon)\| \le \frac{M}{|\lambda|}$, for all $\lambda \in \Sigma \setminus \{0\}$.

Translating these conclusions back into statements about $\mathcal{A}$ itself, we obtain

$$\|\mathcal{R}(\lambda : \mathcal{A})\| = \|\mathcal{R}(\lambda - (\epsilon+\omega) : \mathcal{A}_\epsilon)\| \le \frac{M}{|\lambda - (\epsilon+\omega)|}, \tag{10.6}$$

whenever $\lambda - (\epsilon + \omega) \in \Sigma \setminus \{0\}$, which holds precisely when $\lambda \in S_{\frac{\pi}{2}+\delta,\epsilon+\omega}$. Therefore, the operators $\mathcal{A} = \mathcal{A}(t)$ are sectorial.

Finally, the mapping $t \mapsto \mathcal{A}(t)$ is Lipschitz, since $\exists\ C$ so that

$$\|\mathcal{A}(t) - \mathcal{A}(s)\|_{\mathcal{B}(H^2(\mathbb{R},\mathbb{C}^2),L^2(\mathbb{R},\mathbb{C}^2)))} \le \|\mathbf{B}(t) - \mathbf{B}(s)\|_{\mathbb{C}^{2\times 2}} = \frac{|g(t) - g(s)|}{2\Omega_g^2} \le \frac{C|t-s)|}{2\Omega_g^2},$$

since $t \mapsto g(t)$ is Lipschitz if $\boldsymbol{\psi}$ is differentiable with respect to t. ∎

11. The essential spectrum of the monodromy operator. In this section we prove the main result, Theorem 4.7, which gives conditions under which $\sigma_{\mathrm{ess}}(\mathcal{M}) = \sigma_{\mathrm{ess}}(\mathcal{M}_\infty)$.

Proof of Theorem 4.7. The lumped model we consider consists of fiber segments (single-mode fibers and a fiber amplifier) and discrete input-output devices (a dispersion compensation element, an output coupler, and a fast saturable absorber). We let $t \in [0, T]$ denote location in the laser loop. In a fiber segment of length, L, that starts at location $t = T_1$, we have $t = t_{\mathrm{loc}} + T_1 \in [T_1, T_1 + L]$, where t_{loc} denotes distance along the fiber. For an input-output device at location, t, we use t_- and t_+ to denote the locations of the input and output to the device, and we impose the ordering $t_- < t_+$. We let $\mathcal{U}(t, s)$ and $\mathcal{U}_\infty(t, s)$, for $t > s$, denote the linearized evolution and the asymptotic linearized evolution operators from location s to location t. In particular, for an input-output device at location, t, the linearized transfer operator of the device is denoted by $\mathcal{U}(t_+, t_-)$. The corresponding monodromy operators are then given by $\mathcal{M} = \mathcal{U}(T, 0)$ and $\mathcal{M}_\infty = \mathcal{U}_\infty(T, 0)$. As in (3.1), $\mathcal{M}$ and $\mathcal{M}_\infty$ are both compositions of the linearized transfer operators of the fibers and devices in the lumped model. By Weyl's essential spectrum theorem [20], we just need to show that there is a compact operator, $\mathcal{K}$ so that

$$\mathcal{M} = \mathcal{M}_\infty + \mathcal{K}. \tag{11.1}$$

To do so we will inductively show that at the location, t, of the end of each fiber segment that

$$\mathcal{U}(t, 0) = \mathcal{U}_\infty(t, 0) + \mathcal{K}(t), \tag{11.2}$$

and that at the exit, t_+, to each input-output device, that

$$\mathcal{U}(t_+, 0) = \mathcal{U}_\infty(t_+, 0) + \mathcal{K}(t_+), \tag{11.3}$$

for some compact operators, $\mathcal{K}(t)$ and $\mathcal{K}(t_+)$.

First, we show that (11.2) holds in the fiber amplifier. The argument is the same for the single-mode fibers. For a fiber segment of length, L, starting at location, T_1, an argument based on the variation of parameters formula (see [39, Lemma 5.1]) shows that, for all $t \in [T_1, T_1 + L]$,

$$\mathcal{U}(t,0) \;=\; \mathcal{U}_\infty(t,T_1) \circ \mathcal{U}(T_1,0) \;+\; \int_{T_1}^{t} \mathcal{U}_\infty(t,t') \circ \mathbf{M}(t') \circ \mathcal{U}(t',0)\, dt', \tag{11.4}$$

where $\mathbf{M}$ is the multiplication operator given by (9.3). Indeed, this equation is consistent at $t = T_1$ and implies that

$$\partial_t \mathcal{U}(t,0) = \mathcal{L}(t)\mathcal{U}(t,0). \tag{11.5}$$

Lemma 11.1. *The operator*

$$\widetilde{\mathcal{K}}(t) = \int_{T_1}^{t} \mathcal{U}_\infty(t,t') \circ \mathbf{M}(t') \circ \mathcal{U}(t',0)\, dt' \tag{11.6}$$

is compact.

Given this lemma and substituting the induction hypothesis,

$$\mathcal{U}(T_1,0) = \mathcal{U}_\infty(T_1,0) + \mathcal{K}(T_1), \tag{11.7}$$

into (11.4) yields (11.2) with

$$\mathcal{K}(t) \;=\; \mathcal{U}_\infty(t,T_1) \circ \mathcal{K}(T_1) \;+\; \widetilde{\mathcal{K}}(t), \tag{11.8}$$

which is compact since the composition of a bounded and a compact operator is compact.

Second, we show that (11.3) holds for each input-output device. Let

$$\mathcal{B}(t_+,t_-) \;=\; \mathcal{U}(t_+,t_-) - \mathcal{U}_\infty(t_+,t_-). \tag{11.9}$$

For all the input-output devices in the lumped model we are considering, except for the fast saturable absorber, $\mathcal{B}(t_+,t_-) = 0$. By (3.6), for the saturable absorber, $\mathcal{B}(t_+,t_-)(\boldsymbol{u}) = \mathbf{B}\boldsymbol{u}$ is a multiplication operator with

$$\mathbf{B}(x) = (\ell_0 - \ell(\psi(x))\,\mathbf{I} - \frac{2\ell^2(\psi(x))}{\ell_0 P_{\mathrm{sat}}} \boldsymbol{\psi}\boldsymbol{\psi}^T, \tag{11.10}$$

where

$$\ell(\psi) = \frac{\ell_0}{1 + |\psi_{\mathrm{in}}|^2/P_{\mathrm{sat}}}. \tag{11.11}$$

Since $\boldsymbol{\psi}$ is assumed to be bounded, $\mathcal{B}(t_+,t_-) \in \mathcal{B}(L^2(\mathbb{R},\mathbb{C}^2))$ is bounded but is not compact. Nevertheless, we have the following theorem.

Theorem 11.2. *Under the assumptions of Theorem 4.7, for the fast saturable absorber the operator, $\mathcal{B}(t_+,t_-) \circ \mathcal{U}_\infty(t_-,0)$, is compact.*

Given this theorem and substituting the induction hypothesis,

$$\mathcal{U}(t_-,0) = \mathcal{U}_\infty(t_-,0) + \mathcal{K}(t_-), \tag{11.12}$$

into $\mathcal{U}(t_+,0) = \mathcal{U}(t_+,t_-) \circ \mathcal{U}(t_-,0)$ yields (11.3) with

$$\mathcal{K}(t_+) \;=\; \mathcal{B}(t_+,t_-) \circ \mathcal{U}_\infty(t_-,0) \;+\; \mathcal{U}(t_+,t_-) \circ \mathcal{K}(t_-), \tag{11.13}$$

which is compact by Theorem 11.2 and Proposition 6.1. ∎

Proof of Lemma 11.1. The proof uses the same basic ideas as in the proof of the analogous result for the complex Ginzburg-Landau equation given in [39, Theorem 5.1]. Here we confine our attention to showing that the integrand, $\mathcal{C}$, in (11.6) is compact. To do so, it suffices to show that the adjoint, $\mathcal{C}^*$, is compact.

Throughout the proof, we use times, $0 < s < t < L$, that are local to the fiber, and we let $\tau = L - t$ and $\sigma = L - s$ be the corresponding backwards time variables. Since the adjoint differential operator is defined by $\mathcal{L}^*(\tau) := [\mathcal{L}(L-\tau)]^*$, we have that

$$\mathcal{L}^*(\tau) = \mathcal{L}^*_\infty(\tau) + \mathbf{M}^*(L-\tau). \tag{11.14}$$

By definition, the adjoint linearized evolution operator, $\mathcal{U}^*(\sigma, \tau)$, in the fiber is the operator that satisfies

$$\partial_\sigma \mathcal{U}^*(\sigma, \tau) = \mathcal{L}^*(\sigma)\mathcal{U}^*(\sigma, \tau). \tag{11.15}$$

This operator is characterized by the equation

$$\langle \mathcal{U}(t,s)\boldsymbol{u}(s), \boldsymbol{v}(\tau)\rangle_{L^2(\mathbb{R},\mathbb{C}^2)} \;=\; \langle \boldsymbol{u}(s), \mathcal{U}^*(\sigma,\tau)\boldsymbol{v}(\tau)\rangle_{L^2(\mathbb{R},\mathbb{C}^2)}. \tag{11.16}$$

Therefore,

$$[\mathcal{U}(t,s)]^* \;=\; \mathcal{U}^*(L-s, L-t). \tag{11.17}$$

Letting $\tau' = L - t'$, we find that

$$\mathcal{C}^* \;=\; \mathcal{U}^*(L,\tau') \circ \mathbf{M}^*(L-\tau') \circ \mathcal{U}^*_\infty(\tau', \tau). \tag{11.18}$$

As in Theorem 9.1, $\mathcal{L}^*(\tau')$ is a relatively compact perturbation of $\mathcal{L}^*_\infty(\tau')$. Therefore, there is a $\lambda(\tau') \in \rho(\mathcal{L}^*_\infty(\tau'))$ so that $\mathbf{M}^*(L-\tau') \circ (\mathcal{L}^*_\infty(\tau') - \lambda(\tau'))^{-1}$ is compact. Furthermore, by Theorem 10.2 for the fiber amplifier (which also holds for the adjoint operators) and the corresponding result for the single mode fibers (modeled with the additional spectral filtering term as in (4.9), see [39, Lemma 5.2]), we have that $(\mathcal{L}^*_\infty(\tau') - \lambda(\tau')) \circ \mathcal{U}^*_\infty(\tau', \tau)$ is bounded. Therefore,

$$\mathcal{C}^* \;=\; \mathcal{U}^*(L,\tau') \circ \mathbf{M}^*(L-\tau') \circ (\mathcal{L}^*_\infty(\tau') - \lambda(\tau'))^{-1} \circ (\mathcal{L}^*_\infty(\tau') - \lambda(\tau')) \circ \mathcal{U}^*_\infty(\tau', \tau). \tag{11.19}$$

is compact, as required. ∎

The proof of Theorem 11.2 relies on the Kolmogorov-Riesz compactness theorem, which can be stated as follows [10].

THEOREM 11.3. *A subset, $\mathfrak{F} \subset L^2(\mathbb{R}, \mathbb{C}^2)$, is totally bounded if and only if the following three conditions hold:*

1. *$\mathfrak{F}$ is bounded,*
2. *for all $\epsilon > 0$ there is an $R > 0$ so that for all $f \in \mathfrak{F}$,*

$$\int_{|x|>R} \|f(x)\|^2_{\mathbb{C}^2}\, dx \;<\; \epsilon^2, \qquad \textit{and} \tag{11.20}$$

3. *for all $\epsilon > 0$ there is a $\delta > 0$ so that for all $f \in \mathfrak{F}$ and $y \in \mathbb{R}$ with $|y| < \delta$,*

$$\int_{\mathbb{R}} \|f(x+y) - f(x)\|^2_{\mathbb{C}^2}\, dx \;<\; \epsilon^2. \tag{11.21}$$

Proof of Theorem 11.2. We first show that, at the input to the saturable absorber,

$$\mathcal{U}_\infty(t_-, 0) \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2), H^2(\mathbb{R}, \mathbb{C}^2)). \tag{11.22}$$

This property holds since the transfer operators for the fiber amplifier and the single-mode fibers with an additional spectral filtering term satisfy

$$\mathcal{U}_\infty^{\mathrm{FA}}, \mathcal{U}_\infty^{\mathrm{SMF}} \in \mathcal{B}(L^2(\mathbb{R}, \mathbb{C}^2), H^2(\mathbb{R}, \mathbb{C}^2)), \tag{11.23}$$

and since (4.12) holds for the DCF element and the output coupler. To establish (11.23) for $\mathcal{U}_\infty^{\mathrm{FA}}$, we use (8.7) to obtain

$$\|\mathcal{U}_\infty^{\mathrm{FA}} \boldsymbol{u}\|^2_{H^2(\mathbb{R},\mathbb{C}^2)} \le C_1 \|(1+\omega^2)(\widehat{\mathcal{U}_\infty^{\mathrm{FA}}} \widehat{\boldsymbol{u}})(\omega)\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \tag{11.24}$$

$$= C_1 \int_{\mathbb{R}} (1+\omega^2)^2 \exp\left((1-\omega^2/\Omega_g^2)G_{\mathrm{FA}}\right) \|\widehat{\boldsymbol{u}}(\omega)\|^2_{\mathbb{C}^2}\, d\omega \tag{11.25}$$

$$\le C_2 \|\boldsymbol{u}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)}. \tag{11.26}$$

The proof for $\mathcal{U}_\infty^{\mathrm{SMF}}$ is similar.

From this point on, the proofs is analogous to the proof of [39, Theorem 3.1] that, for the complex Ginzburg-Landau equation, $\mathcal{L}(t)$ is a relatively compact perturbation of $\mathcal{L}_\infty$ There we showed that the operator $\mathbf{M}(t) \circ (\mathcal{L}_\infty - \lambda)^{-1}$ was compact using the exponential decay and weak regularity of $\boldsymbol{\psi}$ and the fact that $(\mathcal{L}_\infty - \lambda)^{-1}$ maps bounded sets in $L^2(\mathbb{R}, \mathbb{C}^2)$ to bounded sets in $H^2(\mathbb{R}, \mathbb{C}^2)$ (endowed with the standard Sobolev norm). Here we show that $\mathcal{K} := \mathcal{B}(t_+, t_-) \circ \mathcal{U}_\infty(t_-, 0)$, is compact using the exponential decay and weak regularity of $\boldsymbol{\psi}$ in the saturable absorber, together with (11.23). Specifically, it suffices to show that for any bounded family of functions, $\mathfrak{H} \subset L^2(\mathbb{R}, \mathbb{C}^2)$, the subset $\mathfrak{F} = \mathcal{K}(\mathfrak{H}) \subset L^2(\mathbb{R}, \mathbb{C}^2)$ is totally bounded. To do so, we check the three conditions of the Kolmogorov-Riesz compactness Theorem 11.3.

For the first condition, we observe that $\mathfrak{F}$ is bounded since the operator $\mathcal{K}$ and the subset $\mathfrak{H}$ are both bounded. Let $\mathfrak{G} = \mathcal{U}_\infty(t_-, 0)(\mathfrak{H}) \subset H^2(\mathbb{R}, \mathbb{C}^2)$. Since $\mathfrak{H}$ is bounded, (11.22) implies that

$$\sup_{g \in \mathfrak{G}} \|g\|_{H^2(\mathbb{R},\mathbb{C}^2)} < \infty. \tag{11.27}$$

To verify the second condition, given $f \in \mathfrak{F}$, there is a $g \in \mathfrak{G}$ so that $f = \mathbf{B} g$ where $\mathbf{B}$ is given by (11.10). Therefore,

$$\int\limits_{|x|>R} \|f(x)\|^2_{\mathbb{C}^2}\, dx \;\le\; \int\limits_{|x|>R} \|\mathbf{B}(x)\|^2_{\mathbb{C}^{2\times 2}} \|g(x)\|^2_{\mathbb{C}^2}\, dx. \tag{11.28}$$

Let $C_{\mathfrak{G}} = \sup\limits_{g \in \mathfrak{G}} \|g\|_{L^2(\mathbb{R},\mathbb{C}^2)}$. By Hypothesis 4.1, $\exists\ R_1 > 0$ so that $\|\mathbf{B}(x)\|_{\mathbb{C}^{2\times 2}} < e^{-r|x|}/C_{\mathfrak{G}}$ for all $|x| > R_1$. Therefore, if $R > R_1$,

$$\int\limits_{|x|>R} \|f(x)\|^2_{\mathbb{C}^2}\, dx \;\le\; \frac{1}{C_{\mathfrak{G}}^2} e^{-2rR} \int_{|x|>R} \|g(x)\|^2_{\mathbb{C}^2}\, dx \;\le\; e^{-2rR} \;\le\; \epsilon^2, \tag{11.29}$$

provided also that $R > |\log \epsilon|/r$.

For the third condition, we recall from Hypothesis 4.1 that $\mathbf{B} \in C^1(\mathbb{R}, \mathbb{C}^{2\times 2})$. Since $\mathfrak{G} \subset H^2(\mathbb{R}, \mathbb{C}^2)$, we know that $\mathfrak{F} \subset H^1(\mathbb{R}, \mathbb{C}^2)$. By a result in Evans [6, §5.8.2] on the difference quotient of a H^1 function, we find that,

$$\begin{aligned}
\int_{\mathbb{R}} \|f(x+y) - f(x)\|_{\mathbb{C}^2}^2 \, dx &\leq |y|^2 \, \|f_x\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 \\
&\leq |y|^2 \, \left[\|\mathbf{B}_x g\|_{L^2(\mathbb{R},\mathbb{C}^2)} + \|\mathbf{B} g_x\|_{L^2(\mathbb{R},\mathbb{C}^2)}\right]^2 \\
&\leq C|y|^2 \max\{\|\mathbf{B}\|_{L^\infty(\mathbb{R},\mathbb{C}^2)}^2, \|\mathbf{B}_x\|_{L^\infty(\mathbb{R},\mathbb{C}^2)}^2\} \, \|g\|_{H^2(\mathbb{R},\mathbb{C}^2)}^2, \qquad (11.30)
\end{aligned}$$

for some constant, C. Finally, by Hypothesis 4.1 and (11.27), the right hand side of (11.30) can be made arbitrarily small, provided y is close enough to zero. ◻

Appendix A. Completion of Proof of Lemma 6.7. To complete the proof we establish the estimates in (6.11) and (6.12). By (6.4), (6.5), and (6.6),

$$\begin{aligned}
&\|F(t+h) - F(t) - hF'(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\leq \left\|\{\mathbf{B}(t+h) - \mathbf{B}(t) - h\partial_t \mathbf{B}(t)\}\partial_x^2 \boldsymbol{v}\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&+ \left\|\{\widetilde{\mathbf{M}}_1(t+h) - \widetilde{\mathbf{M}}_1(t) - \partial_t \widetilde{\mathbf{M}}_1(t)\}\boldsymbol{v}\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&+ \|\boldsymbol{\phi}(t+h)\langle \boldsymbol{\psi}(t+h), \boldsymbol{v}\rangle - \boldsymbol{\phi}(t)\langle \boldsymbol{\psi}(t), \boldsymbol{v}\rangle - h\partial_t(\boldsymbol{\phi}(t)\langle \boldsymbol{\psi}(t), \boldsymbol{v}\rangle)\|_{L^2(\mathbb{R},\mathbb{C}^2)}.
\end{aligned} \qquad (A.1)$$

To establish (6.11) we estimate each of the term in (A.1). We estimate the first term in (A.1) by

$$\begin{aligned}
&\left\|\{\mathbf{B}(t+h) - \mathbf{B}(t) - h\partial_t \mathbf{B}(t)\}\partial_x^2 \boldsymbol{v}\right\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 \\
\leq\ & \|\mathbf{B}(t+h) - \mathbf{B}(t) - h\partial_t \mathbf{B}(t)\|_{\mathbb{C}^{2\times 2}}^2 \int_{\mathbb{R}} \left\|\partial_x^2 \boldsymbol{v}(x)\right\|_{\mathbb{C}^2}^2 dx \\
\leq\ & \left\|\int_t^{t+h} \{(\partial_t \mathbf{B})(\tau) - (\partial_t \mathbf{B})(t)\} d\tau\right\|_F^2 \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)}^2 \\
=\ & \sum_{i,j=1}^{2} \left|\int_t^{t+h} \{(\partial_t \mathbf{B})_{ij}(\tau) - (\partial_t \mathbf{B})_{ij}(t)\} d\tau\right|^2 \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)}^2 \\
\leq\ & \sum_{i,j=1}^{2} h \int_t^{t+h} |(\partial_t \mathbf{B})_{ij}(\tau) - (\partial_t \mathbf{B})_{ij}(t)|^2 d\tau \quad \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)}^2 ,
\end{aligned}$$

where the last inequality follows from

$$\left|\int_a^b f(\tau) d\tau\right|^2 \leq (b-a) \int_a^b |f(\tau)|^2 d\tau, \qquad (A.2)$$

which is a special case of the Cauchy-Schwarz inequality. Consequently,

$$\begin{aligned}
&\left\|\{\mathbf{B}(t+h) - \mathbf{B}(t) - h\partial_t \mathbf{B}(t)\}\partial_x^2 \boldsymbol{v}\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\leq 2\sqrt{2}h \sup_{\tau \in (t,t+h)} \|(\partial_t \mathbf{B})(\tau) - (\partial_t \mathbf{B})(t)\|_{\mathbb{C}^{2\times 2}} \quad \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)} .
\end{aligned} \qquad (A.3)$$

Performing a similar calculation to estimate the second term in (A.1), we obtain

$$
\begin{aligned}
&\left\|\{\widetilde{\mathbf{M}}_1(t+h) - \widetilde{\mathbf{M}}_1(t) - h\partial_t\widetilde{\mathbf{M}}_1(t)\}\boldsymbol{v}\right\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\le \int_{\mathbb{R}} \left\|\int_t^{t+h}\{(\partial_t\widetilde{\mathbf{M}}_1)(\tau,x) - (\partial_t\widetilde{\mathbf{M}}_1)(t,x)\}d\tau\right\|^2_{\mathbb{C}^{2\times 2}} \|\boldsymbol{v}(x)\|^2_{\mathbb{C}^2}\,dx \\
&\le \sup_{x\in\mathbb{R}} \left\|\int_t^{t+h}\{(\partial_t\widetilde{\mathbf{M}}_1)(\tau,x) - (\partial_t\widetilde{\mathbf{M}}_1)(t,x)\}d\tau\right\|^2_F \|\boldsymbol{v}\|^2_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\le \sup_{x\in\mathbb{R}} \sum_{i,j=1}^2 \left|\int_t^{t+h}\left\{(\partial_t\widetilde{\mathbf{M}}_1)_{ij}(\tau,x) - (\partial_t\widetilde{\mathbf{M}}_1)_{ij}(t,x)\right\}d\tau\right|^2 \quad \|\boldsymbol{v}\|^2_{H^2(\mathbb{R},\mathbb{C}^2)} \\
&\le \sup_{x\in\mathbb{R}} \sum_{i,j=1}^2 h\int_t^{t+h}\left|(\partial_t\widetilde{\mathbf{M}}_1)_{ij}(\tau,x) - (\partial_t\widetilde{\mathbf{M}}_1)_{ij}(t,x)\right|^2 d\tau \quad \|\boldsymbol{v}\|^2_{H^2(\mathbb{R},\mathbb{C}^2)} \\
&\le h^2 \sup_{(\tau,x)\in(t,t+h)\times\mathbb{R}} \left\|(\partial_t\widetilde{\mathbf{M}}_1)(\tau,x) - (\partial_t\widetilde{\mathbf{M}}_1)(t,x)\right\|^2_F \quad \|\boldsymbol{v}\|^2_{H^2(\mathbb{R},\mathbb{C}^2)} \\
&\le 8h^2 \sup_{(\tau,x)\in(t,t+h)\times\mathbb{R}} \left\|(\partial_t\widetilde{\mathbf{M}}_1)(\tau,x) - (\partial_t\widetilde{\mathbf{M}}_1)(t,x)\right\|^2_{\mathbb{C}^{2\times 2}} \quad \|\boldsymbol{v}\|^2_{H^2(\mathbb{R},\mathbb{C}^2)}\,.
\end{aligned}
\tag{A.4}
$$

Next, adding and subtracting $\boldsymbol{\phi}(t+h)\langle\boldsymbol{\psi}(t),\boldsymbol{v}\rangle$ in the third term of (A.1), we obtain

$$
\begin{aligned}
&\|\boldsymbol{\phi}(t+h)\langle\boldsymbol{\psi}(t+h),\boldsymbol{v}\rangle - \boldsymbol{\phi}(t)\langle\boldsymbol{\psi}(t),\boldsymbol{v}\rangle - h\partial_t(\boldsymbol{\phi}(t)\langle\boldsymbol{\psi}(t),\boldsymbol{v}\rangle)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\le \|\boldsymbol{\phi}(t+h)\langle\boldsymbol{\psi}(t+h) - \boldsymbol{\psi}(t),\boldsymbol{v}\rangle - \boldsymbol{\phi}(t)\langle h\partial_t\boldsymbol{\psi}(t),\boldsymbol{v}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&+ \|\{\boldsymbol{\phi}(t+h) - \boldsymbol{\phi}(t) - h\partial_t\boldsymbol{\phi}(t)\}\langle\boldsymbol{\psi}(t),\boldsymbol{v}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)}\,.
\end{aligned}
\tag{A.5}
$$

Now, for any $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w} \in L^2(\mathbb{R},\mathbb{C}^2)$,

$$
\|\boldsymbol{u}\langle\boldsymbol{v},\boldsymbol{w}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)} \le \|\boldsymbol{u}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \ \|\boldsymbol{v}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \ \|\boldsymbol{w}\|_{L^2(\mathbb{R},\mathbb{C}^2)}\,.
\tag{A.6}
$$

To estimate the first term in (A.5), we add and subtract $\boldsymbol{\phi}(t+h)\langle h\partial_t\boldsymbol{\psi}(t),\boldsymbol{v}\rangle$ and use (A.6) to obtain

$$
\begin{aligned}
&\|\boldsymbol{\phi}(t+h)\langle\boldsymbol{\psi}(t+h) - \boldsymbol{\psi}(t),\boldsymbol{v}\rangle - \boldsymbol{\phi}(t)\langle h\partial_t\boldsymbol{\psi}(t),\boldsymbol{v}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\le \Big\{ \|\boldsymbol{\phi}(t+h)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \|\boldsymbol{\psi}(t+h) - \boldsymbol{\psi}(t) - h\partial_t\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\quad + \|\boldsymbol{\phi}(t+h) - \boldsymbol{\phi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \|h\partial_t\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \Big\} \|\boldsymbol{v}\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&= \Big\{ \|\boldsymbol{\phi}(t+h)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \left\|\int_t^{t+h}\{(\partial_t\boldsymbol{\psi})(\tau) - (\partial_t\boldsymbol{\psi})(t)\}d\tau\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\quad + h\left\|\int_t^{t+h}(\partial_t\phi)(\tau)d\tau\right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \|\partial_t\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \Big\} \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)} \\
&\le \Big\{ h\,\|\boldsymbol{\phi}(t+h)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \sup_{\tau\in(t,t+h)} \|(\partial_t\boldsymbol{\psi})(\tau) - (\partial_t\boldsymbol{\psi})(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\
&\quad + h^2 \sup_{\tau\in(t,t+h)} \|(\partial_t\boldsymbol{\phi})(\tau)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \quad \|\partial_t\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \Big\} \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)}\,.
\end{aligned}
\tag{A.7}
$$

Now,

$$\|\boldsymbol{\phi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \le \frac{g_0 C}{E_{\text{sat}}} \|\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}, \tag{A.8}$$

and

$$\begin{aligned}\|\partial_t \boldsymbol{\phi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \le& \frac{1}{g_0 E_{\text{sat}}} \left\{ \left| \frac{-2}{E_{\text{sat}}} g^3(t) E'(t) \right| \; \left\| \left(\boldsymbol{\psi}(t) + \frac{\partial_x^2 \boldsymbol{\psi}(t)}{\Omega_g^2} \right) \right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \right. \\ & \left. + g^2(t) \left\| \partial_t \left(\boldsymbol{\psi}(t) + \frac{\partial_x^2 \boldsymbol{\psi}(t)}{\Omega_g^2} \right) \right\|_{L^2(\mathbb{R},\mathbb{C}^2)} \right\} \\ \le& \frac{2 g_0^2 C}{E_{\text{sat}}^2} |E'(t)| \; \|\boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)} + \frac{2 g_0 C}{E_{\text{sat}}} \|\partial_t \boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)}.\end{aligned} \tag{A.9}$$

Substituting (A.8) and (A.9) in (A.7), we obtain

$$\begin{aligned}& \|\boldsymbol{\phi}(t+h)\langle \boldsymbol{\psi}(t+h) - \boldsymbol{\psi}(t), \boldsymbol{v}\rangle - \boldsymbol{\phi}(t)\langle h \partial_t \boldsymbol{\psi}(t), \boldsymbol{v}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\ \le & \left\{ \frac{g_0 h C}{E_{\text{sat}}} \|\boldsymbol{\psi}(t+h)\|_{H^2(\mathbb{R},\mathbb{C}^2)} \sup_{\tau \in (t,t+h)} \|(\partial_t \boldsymbol{\psi})(\tau) - (\partial_t \boldsymbol{\psi})(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)} \right. \\ & + \frac{2 g_0^2 h^2 C}{E_{\text{sat}}^2} \sup_{\tau \in (t,t+h)} |E'(\tau)| \, \|\boldsymbol{\psi}(\tau)\|_{H^2(\mathbb{R},\mathbb{C}^2)} \|\partial_t \boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)} \\ & \left. + \frac{2 g_0 h^2 C}{E_{\text{sat}}} \sup_{\tau \in (t,t+h)} \|\partial_t \boldsymbol{\psi}(\tau)\|_{H^2(\mathbb{R},\mathbb{C}^2)} \|\partial_t \boldsymbol{\psi}(t)\|_{H^2(\mathbb{R},\mathbb{C}^2)} \right\} \|\boldsymbol{v}\|_{H^2(\mathbb{R},\mathbb{C}^2)}.\end{aligned} \tag{A.10}$$

Next to estimate the second term in (A.5) we use (A.6) to obtain

$$\begin{aligned}& \|\{\boldsymbol{\phi}(t+h) - \boldsymbol{\phi}(t) - h \partial_t \boldsymbol{\phi}(t)\} \langle \boldsymbol{\psi}(t), \boldsymbol{v}\rangle\|_{L^2(\mathbb{R},\mathbb{C}^2)} \\ \le & \|\boldsymbol{\phi}(t+h) - \boldsymbol{\phi}(t) - h \partial_t \boldsymbol{\phi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \|\boldsymbol{\psi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)} \|\boldsymbol{v}\|_{L^2(\mathbb{R},\mathbb{C}^2)},\end{aligned} \tag{A.11}$$

and observe that, by (A.2) and Fubini's theorem,

$$\begin{aligned}\|\boldsymbol{\phi}(t+h) - \boldsymbol{\phi}(t) - h \partial_t \boldsymbol{\phi}(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 =& \left\| \int_t^{t+h} ((\partial_t \boldsymbol{\phi})(\tau) - (\partial_t \boldsymbol{\phi})(t)) \, d\tau \right\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 \\ \le& h \int_t^{t+h} \|(\partial_t \boldsymbol{\phi})(\tau) - (\partial_t \boldsymbol{\phi})(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2 \, d\tau \\ \le& h^2 \sup_{\tau \in (t,t+h)} \|(\partial_t \boldsymbol{\phi})(\tau) - (\partial_t \boldsymbol{\phi})(t)\|_{L^2(\mathbb{R},\mathbb{C}^2)}^2.\end{aligned} \tag{A.12}$$

Finally, substituting (A.3), (A.4), (A.11), and (A.12) in (A.1), we obtain (6.11).

Acknowledgments. We thank Yuri Latushkin and Curtis Menyuk for generously sharing their expertise with us. We thank the anonymous reviewers for their suggestions to improve the paper.

REFERENCES

[1] G. P. Agrawal, *Nonlinear fiber optics*, Academic press, 2007.

[2] D. M. Ambrose and J. Wilkening, *Computing time-periodic solutions of nonlinear systems of partial differential equations*, in Hyperbolic Problems: Theory, Numerics and Applications (In 2 Volumes), World Scientific, 2012, pp. 273–280.
[3] S. A. Diddams, *The evolving optical frequency comb*, J. Opt. Soc. Am. B, 27 (2010), pp. B51–B62.
[4] D. E. Edmunds and W. D. Evans, *Spectral theory and differential operators*, Oxford University Press, 2018.
[5] K.-J. Engel and R. Nagel, *One-parameter semigroups for linear evolution equations*, Springer Science+Business Media, New York, 2000.
[6] L. Evans, *Partial Differential Equations*, American Mathematical Society, Providence, RI, 2010.
[7] M. E. Fermann, V. I. Kruglov, B. C. Thomsen, J. M. Dudley, and J. D. Harvey, *Self-similar propagation and amplification of parabolic pulses in optical fibers*, Phys. Rev. Lett., 84 (2000), pp. 6010–6013.
[8] W. Fu, L. G. Wright, P. Sidorenko, S. Backus, and F. W. Wise, *Several new directions for ultrafast fiber lasers*, Opt. Express, 26 (2018), pp. 9432–9463.
[9] C. R. Giles and E. Desurvire, *Modeling erbium-doped fiber amplifiers*, Journal of Lightwave Technology, 9 (1991), pp. 271–283.
[10] H. Hanche-Olsen and H. Holden, *The Kolmogorov–Riesz compactness theorem*, Expositiones Mathematicae, 28 (2010), pp. 385–394.
[11] I. Hartl, T. R. Schibli, A. Marcinkevicius, D. C. Yost, D. D. Hudson, M. E. Fermann, and J. Ye, *Cavity-enhanced similariton Yb-fiber laser frequency comb:* 3×10^{14} *W/cm*^{2} *peak intensity at 136 MHz*, Opt. Lett., 32 (2007), pp. 2870–2872.
[12] H. A. Haus, *Theory of mode locking with a fast saturable absorber*, Journal of Applied Physics, 46 (1975), pp. 3049–3058.
[13] H. A. Haus, *Mode locking of lasers*, IEEE J. Sel. Top. Quantum Electron., 6 (2000), pp. 1173–1185.
[14] F. Jones, *Lebesgue integration on Euclidean space*, Jones & Bartlett Learning, 2001.
[15] T. Kapitula, J. N. Kutz, and B. Sandstede, *Stability of pulses in the master mode-locking equation*, J. Opt. Soc. Am. B, 19 (2002), pp. 740–746.
[16] T. Kapitula, N. Kutz, and B. Sandstede, *The Evans function for nonlocal equations*, Indiana University Mathematics Journal, 53 (2004), pp. 1095–1126.
[17] T. Kapitula and K. Promislow, *Spectral and dynamical stability of nonlinear waves*, vol. 457, Springer, 2013.
[18] T. Kapitula and B. Sandstede, *Instability mechanism for bright solitary-wave solutions to the cubic-quintic Ginzburg-Landau equation*, J. Opt. Soc. Am. B, 15 (1998), pp. 2757–2762.
[19] T. Kapitula and B. Sandstede, *Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations*, Physica D: Nonlinear Phenomena, 124 (1998), pp. 58–103.
[20] T. Kato, *Perturbation theory for linear operators*, vol. 132, Springer Science & Business Media, 2013.
[21] D. Kaup, *Perturbation theory for solitons in optical fibers*, Physical Review A, 42 (1990), pp. 5689–5694.
[22] H. Kim, P. Qin, Y. Song, H. Yang, J. Shin, C. Kim, K. Jung, C. Wang, and J. Kim, *Sub-20-attosecond timing jitter mode-locked fiber lasers*, IEEE J. Sel. Top. Quantum Electron, 20 (2014), pp. 260–267.
[23] J. N. Kutz, *Mode-locked soliton lasers*, SIAM review, 48 (2006), pp. 629–678.
[24] A. Lunardi, *Analytic semigroups and optimal regularity in parabolic problems*, Springer Science & Business Media, 2012.
[25] C. R. Menyuk and S. Wang, *Spectral methods for determining the stability and noise performance of passively modelocked lasers*, Nanophotonics, 5 (2016), pp. 332–350.
[26] L. F. Mollenauer and R. H. Stolen, *The soliton laser*, Opt. Lett., 9 (1984), pp. 13–15.
[27] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, vol. 44, Springer Science & Business Media, 2012.
[28] M. Rochette, L. R. Chen, K. Sun, and J. Hernandez-Cordero, *Multiwavelength and tunable self-pulsating fiber cavity based on regenerative SPM spectral broadening and filtering*, IEEE Photonics Technology Letters, 20 (2008), pp. 1497–1499.
[29] Y. Shen, J. Zweck, S. Wang, and C. R. Menyuk, *Spectra of short pulse solutions of the cubic–quintic complex Ginzburg–Landau equation near zero dispersion*, Stud. Appl. Math., 137 (2016), pp. 238–255.
[30] V. Shinglot, J. Zweck, and C. Menyuk, *The continuous spectrum of periodically stationary pulses in a stretched pulse laser*, Opt. Lett., 47 (2022), pp. 1490–1493.
[31] P. Sidorenko, W. Fu, L. G. Wright, M. Olivier, and F. W. Wise, *Self-seeded, multi-*

megawatt, Mamyshev oscillator, Opts. Express, 43 (2018), pp. 2672–2675.
[32] K. Tamura, E. P. Ippen, H. A. Haus, and L. E. Nelson, *77-fs pulse generation from a stretched-pulse mode-locked all-fiber ring laser*, Opt. Lett., 18 (1993), pp. 1080–1082.
[33] N. Tarasov, A. M. Perego, D. V. Churkin, K. Staliunas, and S. K. Turitsyn, *Mode-locking via dissipative Faraday instability*, Nature Communications, 7 (2016), pp. 1–5.
[34] G. Teschl, *Ordinary differential equations and dynamical systems*, vol. 140, American Mathematical Soc., 2012.
[35] S. K. Turitsyn, B. G. Bale, and M. P. Fedoruk, *Dispersion-managed solitons in fibre systems and lasers*, Physics reports, 521 (2012), pp. 135–203.
[36] S. Wang, A. Docherty, B. S. Marks, and C. R. Menyuk, *Boundary tracking algorithms for determining the stability of mode-locked pulses*, J. Opt. Soc. Am. B, 31 (2014), pp. 2914–2930.
[37] S. Wang, B. S. Marks, and C. R. Menyuk, *Comparison of models of fast saturable absorption in passively modelocked lasers*, Opts. Express, 24 (2016), pp. 20228–20244.
[38] J. Yang, *Nonlinear waves in integrable and nonintegrable systems*, SIAM, 2010.
[39] J. Zweck, Y. Latushkin, J. Marzuola, and C. Jones, *The essential spectrum of periodically stationary solutions of the complex Ginzburg-Landau equation*, J. Evol. Equ., 21 (2021), pp. 3313–3329.