Euclidean Invariant Computation of Salient Closed Contours in Images

by

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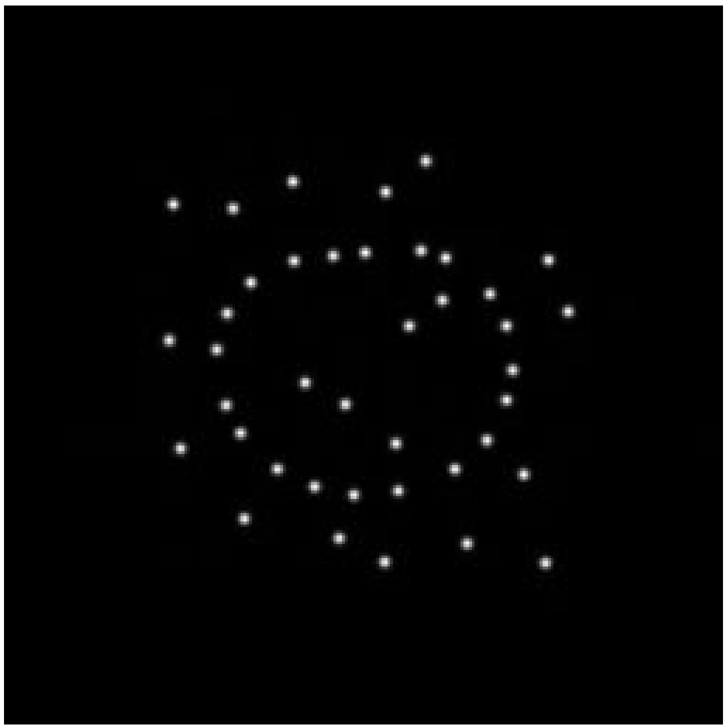
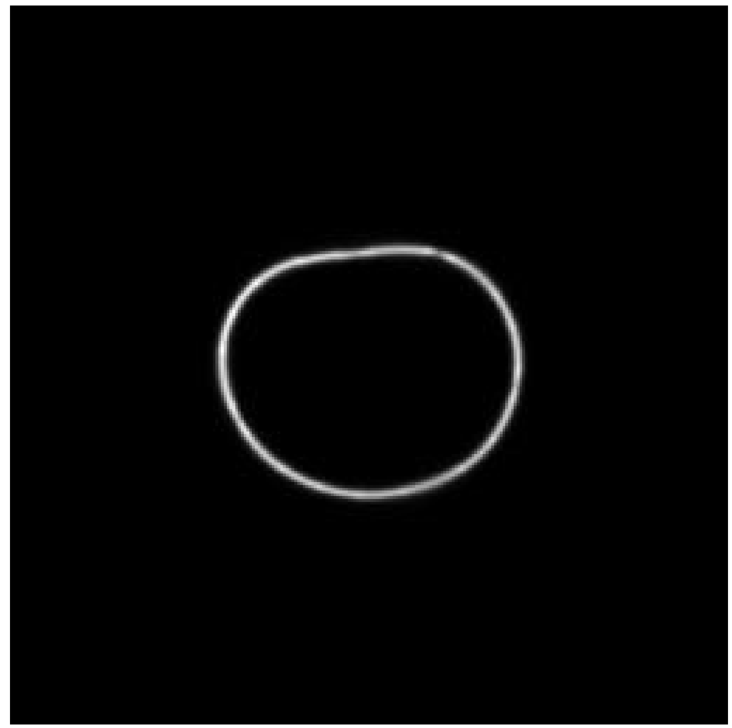
The Contour Completion and Saliency Problem

Compute the most salient closed curve in an image consisting of spots

Input Spot Image	Stochastic completion field
$b : \mathbf{R}^2 \rightarrow \mathbf{R}$	$c : \mathbf{R}^2 \times S^1 \rightarrow \mathbf{R}$
	

$c(\vec{x}, \theta)$ = Probability a closed curve thru spots goes thru $(\vec{x}, \theta)$

Properties of computation

All functions are represented as $f = \sum_{n=1}^{N} c_n \Psi_n$

The computation:

- Acts on the coefficients c_n
- Is implemented in a finite discrete neural network
- Is based on a prior distribution of closed boundary contours
- Is Euclidean invariant in the continuum

Prior probability distribution of boundary contours

- Boundary contours are characterized by a random walk on $\mathbf{R}^2 \times S^1$
  - Particles move at constant speed in $\mathbf{R}^2$ in direction θ
  - Direction θ undergoes a Brownian motion on S^1
- $P(\vec{x}, \theta\,;\,t) \;=\;$ Probability that a particle is at $(\vec{x}, \theta)$ at time t
- Fokker-Planck equation [Mumford, 1994]:

$$\frac{\partial P}{\partial t} \;=\; -\cos\theta\,\frac{\partial P}{\partial x} \;-\; \sin\theta\,\frac{\partial P}{\partial y} \;+\; \frac{\sigma^2}{2}\frac{\partial^2 P}{\partial \theta^2} \;-\; \frac{1}{\tau}P,$$

- Probability particle goes from $(\vec{x}, \theta)$ to $(\vec{y}, \phi)$:

$$P((\vec{y}, \phi) \leftarrow (\vec{x}, \theta)) \;=\; \int_0^\infty P(\vec{y}, \phi\,;t)\,dt, \quad P(\vec{y}, \phi\,;0) = \delta_{(\vec{x},\theta)}(\vec{y}, \phi)$$

Constraining the prior by a spot image

- Model input as finite collection of edges i:

$$b_i(\vec{x}, \theta) \;=\; \text{Prob(Edge } i \text{ is located at } (\vec{x}, \theta))$$

- Initially, no preferred orientations: $b_i(\vec{x}, \theta) \;=\; b_i(\vec{x})$
- Assume edges are distinguishable: $b_i(\vec{x})\, b_j(\vec{x}) \;=\; 0$
- $b(\vec{x}) \;=\; \sum_i b_i(\vec{x}) \;=\;$ Intensity of spot image

Markov process with states i and transition probabilities $P(j \leftarrow i)$:

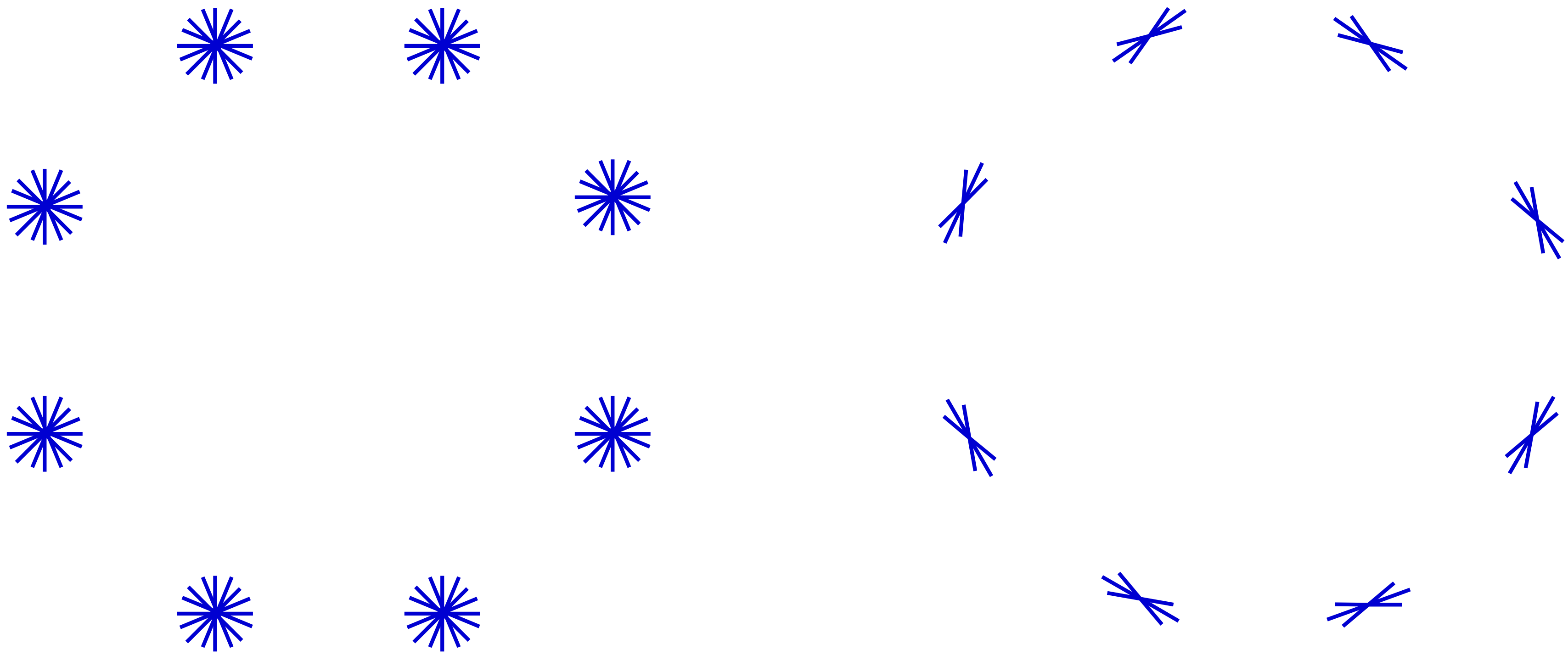
$$P(j \leftarrow i) \;=\; \iint b_j(y)\, P(y \leftarrow x)\, b_i(x)\, dx\, dy, \qquad x \;=\; (\vec{x}, \theta)$$

The eigensource and eigensink fields

- Let Q be the integral linear operator on $L^2(\mathbf{R}^2 \times S^1)$ with kernel

$$Q(y, x) \;=\; b^{1/2}(y)\, P(y \leftarrow x)\, b^{1/2}(x)$$

- Let s and $\overline{s}$ be eigenfunctions: $Qs \;=\; \lambda_{\mathsf{max}} s, \quad \overline{s}\, Q \;=\; \lambda_{\mathsf{max}} \overline{s}$



The stochastic completion field

- $c(\vec{x},\theta) \;=\;$ Probability a particle starting at $(\vec{x},\theta)$ returns to $(\vec{x},\theta)$ after passing through a subset of the edges i
- [Williams and Thornber, 2001] By Perron–Frobenius Theorem:

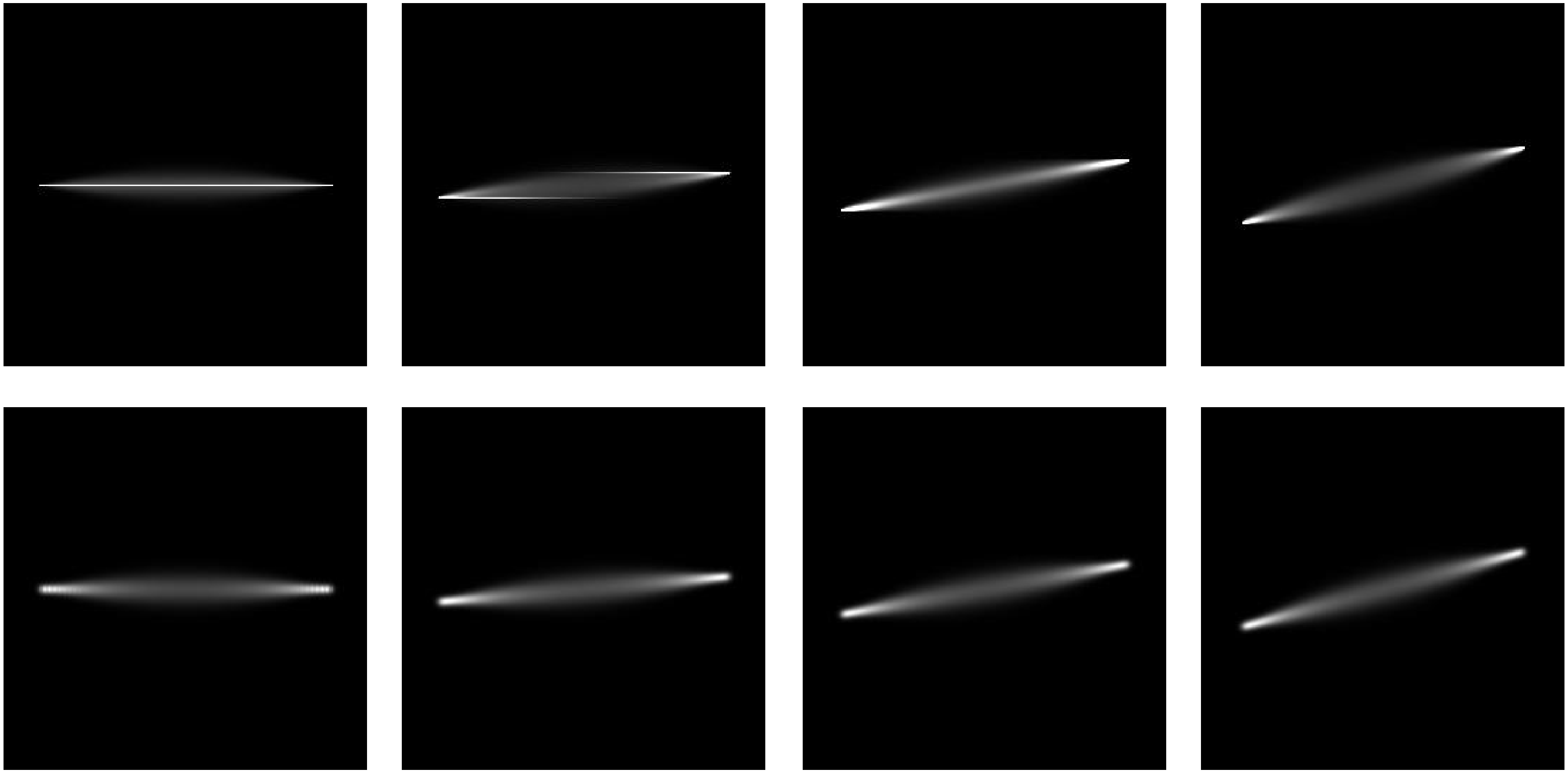
$$c(\vec{x},\theta) \;=\; \frac{p(\vec{x},\theta)\,\overline{p}\,(\vec{x},\theta)}{\lambda_{\mathsf{max}} \int s(\vec{y},\phi)\overline{s}(\vec{y},\phi)\, d\vec{y} d\phi}$$

where the source and sink fields are

$$p \;=\; P b^{1/2} s \quad \text{and} \quad \overline{p} \;=\; \overline{s}\, b^{1/2} P, \qquad \text{with } P(y,x) = P(y \leftarrow x)$$

We want a
biologically-plausible Euclidean-invariant computation of c

Two Point Completion Fields



Standard finite difference scheme in Dirac basis is *not* Euclidean invariant

Shiftable-twistable bases

- Shift-twist transformation on $\mathbf{R}^2 \times S^1$:

$$T_{\vec{x}_0,\theta_0}(\vec{x},\theta) \;=\; (R_{\theta_0}(\vec{x}-\vec{x}_0)\,,\,\theta-\theta_0)$$

- A computation C on $\mathbf{R}^2 \times S^1$ is shift-twist invariant if:

$$\begin{array}{ccc} b & \xrightarrow{C} & c \\ {\scriptstyle T_{\vec{x}_0,\theta_0}}\downarrow & & \downarrow{\scriptstyle T_{\vec{x}_0,\theta_0}} \\ T_{\vec{x}_0,\theta_0}b & \xrightarrow{C} & T_{\vec{x}_0,\theta_0}c \end{array}$$

- A finite set $\mathfrak{F}$ of functions on $\mathbf{R}^2 \times S^1$ forms a shiftable-twistable basis if, for all $(\vec{x}_0,\theta_0) \in \mathbf{R}^2 \times S^1$,

$$T_{\vec{x}_0,\theta_0}(\mathrm{Span}\,\mathfrak{F}) \;\subset\; \mathrm{Span}\,\mathfrak{F}$$

The Gaussian-Fourier basis

- A periodic function $\Psi : [0, X]^2 \times S^1$ is shiftable-twistable if

$$T_{\vec{x}_0,\theta_0}\Psi \;=\; \sum_{\vec{k},m} b_{\vec{k},m}(\vec{x}_0,\theta_0) T_{\vec{k}\Delta, m\Delta_\theta}\Psi$$

- Bandlimited functions are shiftable-twistable [Simoncelli *et al.*, 1992]

- The Gaussian-Fourier shiftable-twistable function:

$$G_\omega(\vec{x},\theta) \;=\; \exp(-\|\vec{x}\|^2/2\nu^2)\exp(i\omega\theta)$$

- The Gaussian-Fourier shiftable-twistable basis:

$$\Psi_{\vec{k},\omega} \;:=\; T_{\vec{k}\Delta,0}G_\omega$$

Euclidean invariant computation of completion fields

- Computation acts on coefficient vectors $\mathbf{s}$ of functions s:

$$s \;=\; \sum \mathbf{s}_{\vec{k},\omega} \Psi_{\vec{k},\omega}$$

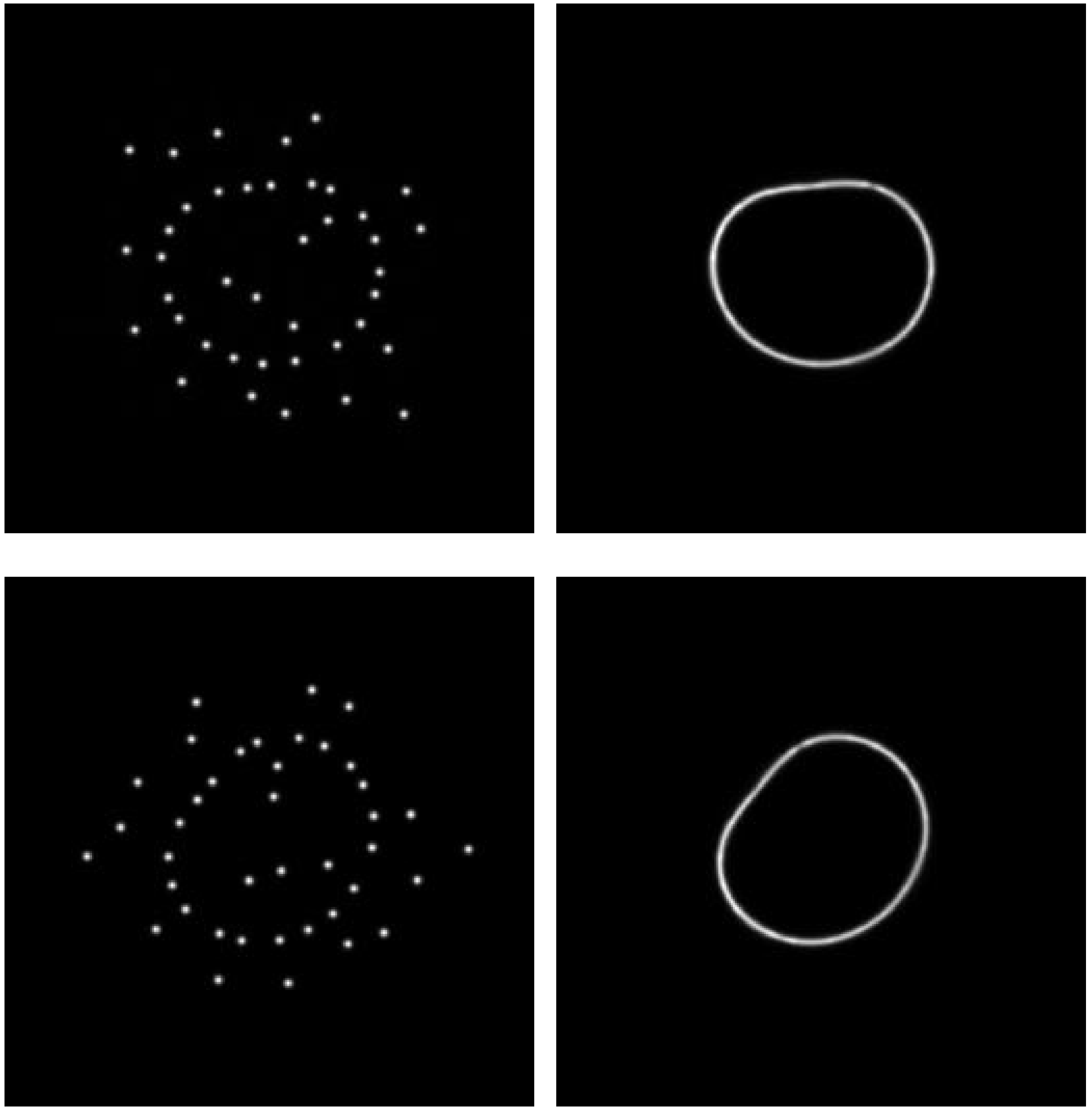
- Represent action of $Q = B^{\frac{1}{2}} P B^{\frac{1}{2}}$ on coefficient vector $\mathbf{s}$ as

$$(\mathbf{Q}\,\mathbf{s})_{\vec{\ell},\eta} \;=\; \sum_{\vec{k},\omega} \mathbf{Q}_{\vec{\ell},\eta;\vec{k},\omega} \mathbf{s}_{\vec{k},\omega}$$

- Power method: $\mathbf{s} = \lim_{m\to\infty} \mathbf{s}^{(m)}$ where $\mathbf{s}^{(m+1)} \;=\; \mathbf{Q}\mathbf{s}^{(m)} / \|s^{(m)}\|$
- Compute $\mathbf{c}$ from $\mathbf{s}$ and synthesize the completion field, c

Demonstration of Euclidean Invariance

$b(\vec{x})$ $\qquad\qquad$ $\int c(\vec{x}, \theta)\, d\theta$



Biological Motivation

Receptive fields in

- Lateral geniculate nucleus are *not* orientation sensitive (spot image)
- Primary visual cortex (V1) are orientation selective (source field)
- Secondary visual cortex complete contours (completion field)

Our computation suggests

Orientation selectivity in V1 is an emergent property of the higher-level computation of salient closed contours

Conclusions

First discrete neural network that computes salient closed contours in images such that

- Input is isotropic, consisting of spots not edges
- Network computes a well-defined function of input
- Based on a distribution of closed contours
- Computation is Euclidean invariant in the continuum

References

- L.R. Williams and J. Zweck, "A rotation and translation invariant discrete saliency network", *Biological Cybernetics*, **88**, (1), pp. 2-10, 2003.
- J. Zweck and L.R. Williams, "Euclidean group invariant computation of stochastic completion fields using shiftable-twistable functions", *Journal of Mathematical Imaging and Vision*, (to appear), 2003.